Beyond GR

Part 1

EE are nice but hard to modify in current form. From purely geometric arguments
$[\nabla_\mu, \nabla_\nu] \to R^\lambda{}_{\mu\rho\nu}$

$$\nabla_{[}{}^{\lambda} R_{\rho}{}^{\nu}{}_{]\lambda\nu} = 0 \quad \to \quad G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu}$$

$$\nabla^\mu G_{\mu\nu} = 0 \quad \text{and} \quad \nabla^\mu T_{\mu\nu} = 0$$

$$\Rightarrow \quad G_{\mu\nu} = 8\pi T_{\mu\nu}$$

Reformulate in a more familiar framework so we can easily adapt it.

Use Lagrangian Formulation. Already extensively studied w/ quantum field theory.

Start with $\quad S = \int \mathcal{L}(q^i, \dot{q}^i)\, d^n x$

$\uparrow$ density

Take scalar gravitational field ϕ

$$\mathcal{L} = -\frac{1}{2}(\partial^\mu \phi)(\partial_\mu \phi) - V(\phi)$$

$$\delta S = \int -(\partial^\mu \delta\phi)(\partial_\mu \phi) - \frac{dV}{d\phi}\delta\phi \; d^n x$$

$$= \int \left[\partial^\mu \partial_\mu \phi - \frac{dV}{d\phi}\right] \delta\phi \, d^n x = 0$$

$$\Rightarrow \quad \partial^2 \phi - \frac{dV}{d\phi} = 0$$

Choose $V = m\phi^2$
this is Klien-Gordon eqn
for scalar field w/ mass m

Part 2

For gravity $\phi \to g_{\mu\nu}$ $\quad S = \int \overset{\text{density}}{\mathcal{L}}(g_{\mu\nu}, \nabla_\sigma g_{\mu\nu}, \dots) d^n x$

$$S = \int \sqrt{-g}\, L(g_{\mu\nu}, \nabla_\sigma g_{\mu\nu}) d^4x$$

$g \neq g^\mu{}_\mu$ $\quad$ density $\quad$ scalar

Which scalar? Keep it simple: R $\quad$ only indep scalar from Riemann tensor

$$S_{EH} = \int \sqrt{-g}\, R\, d^4x \qquad \text{Einstein-Hilbert Action}$$

Vary w/ respect to $g^{\mu\nu}$ inverse metric. Need

$$\delta S_{EH} = \int \left[(\delta\sqrt{-g}) R + \sqrt{-g}(\delta g^{\mu\nu}) R_{\mu\nu} + \sqrt{-g}\, g^{\mu\nu}(\delta R_{\mu\nu}) \right] d^4x$$

easy!

From lin algebra $\quad \ln(\det A) = \mathrm{Tr}(\ln A)$

square matrix A non-zero det $\quad \Rightarrow \frac{1}{\det A} \delta \det A = \mathrm{Tr}(A^{-1}\delta A)$

$$\Rightarrow \quad \delta g = g\,(g^{\mu\nu}\delta g_{\mu\nu})$$

To change $\delta g_{\mu\nu}$ to $\delta g^{\mu\nu}$ take $\delta(g^{\mu\lambda} g_{\lambda\nu}) = \delta(\delta^\mu{}_\nu)$

$$g_{\lambda\nu}\delta g^{\mu\lambda} = -g^{\mu\lambda}\delta g_{\lambda\nu}$$

$$\Rightarrow \delta g_{\mu\nu} = -g_{\mu\rho} g_{\nu\lambda} \delta g^{\rho\lambda}$$

$$\Rightarrow \delta g = -g\,(g_{\mu\nu}\delta g^{\mu\nu})$$

nice!

$$\delta\sqrt{-g} = \frac{1}{2\sqrt{-g}}\delta g = -\frac{1}{2}\sqrt{-g}\, g_{\mu\nu}\delta g^{\mu\nu}$$

Now $\delta R_{\mu\nu} = \delta R^{\lambda}{}_{\mu\lambda\nu}$
$$= \delta\left[\partial_\lambda \Gamma^{\lambda}_{\mu\nu} - \partial_\nu \Gamma^{\lambda}_{\mu\lambda} + \Gamma^{\lambda}_{\lambda\sigma}\Gamma^{\sigma}_{\mu\nu} - \Gamma^{\lambda}_{\nu\sigma}\Gamma^{\sigma}_{\mu\lambda}\right]$$

Lots of work & tedious. Done in Carroll ch 4.

Turns out $\delta R_{\mu\nu} = 0$

Now we have $\delta S_{EH} = \int \sqrt{-g}\left[R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu}\right] \delta g^{\mu\nu} d^n x$

Setting $\delta S_{EH} = 0$ gives $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 0$!!

For matter, add S_M to S_{EH}

$$\delta S_{EH} + \delta S_M = 0$$

$$\Rightarrow \quad G_{\mu\nu} + \frac{1}{\sqrt{-g}} \frac{\delta S_M}{\delta g^{\mu\nu}} = 0$$

Define $T_{\mu\nu} = -8\pi \frac{1}{\sqrt{-g}} \frac{\delta S_M}{\delta g^{\mu\nu}}$
has all properties we want.

Given a metric, can find solution to EE by solving for $T_{\mu\nu}$ (may not be physical though...)

Could have varies S_{EH} WRT $g^{\mu\nu}$ and ∇^μ, get same result plus $\nabla^\mu g_{\mu\nu} = 0$.
This is Palatini formulation.

Popular beyond GR Theories:

1970s • f(R) gravity
1971 • Gauss - Bonnet } Tensor (the metric)
1961 • Brans - Dicke - Scalar tensor (scalar field)
1922 • Einstein - Cartan - non-metric

All beyond GR theories must pass the 3 tests of GR to be considered:
- Perihelion precession of mercury
- Gravitational redshift
- Gravitational lensing

f(R) Gravity

$$S = \int \sqrt{-g}\, f(R)\, d^4x$$

$$\delta S = \int \sqrt{-g}\left[-\frac{1}{2} g_{\mu\nu}\,\delta g^{\mu\nu} f(R) + \frac{df}{dR}\,\delta R\right] d^4x$$

$$\delta R = R_{\mu\nu}\,\delta g^{\mu\nu} + g_{\mu\nu}\,\Box\,\delta g^{\mu\nu} - \nabla_\mu \nabla_\nu g^{\mu\nu}$$

$$\Rightarrow \delta S = \int \sqrt{-g}\left[-\frac{1}{2} g_{\mu\nu}\,\delta g^{\mu\nu} f(R) + \frac{df}{dR}\left(R_{\mu\nu}\,\delta g^{\mu\nu} + g_{\mu\nu}\,\Box\,\delta g^{\mu\nu} - \nabla_\mu \nabla_\nu g^{\mu\nu}\right)\right] d^4x$$

integrate by parts

$$= \int \sqrt{-g}\left[-\frac{1}{2} g_{\mu\nu} f(R) + \frac{df}{dR} R_{\mu\nu} + \left(g_{\mu\nu}\Box - \nabla_\mu \nabla_\nu\right)\left(\frac{df}{dR}\right)\right] \delta g^{\mu\nu}\, d^4x$$

$$\Rightarrow \left(g_{\mu\nu}\Box - \nabla_\mu \nabla_\nu\right)\left(\frac{df}{dR}\right) + \frac{df}{dR} R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} f(R) = 0$$

if $f(R) = R$, this is $G_{\mu\nu} = 0$

not much to say overall

reformulate w/ scalar field

Have box acting on $\frac{df}{dR}$, relate $\frac{df}{dR}$ to scalar field ϕ

This leads to 3rd polarization of $h_{\mu\nu}$, a massive polarization, longitudinal, $\Lambda \sim 10^{-52}$

$f(R) = \Lambda \cdot R + \alpha R^2 + \dots$ α must also be small $\sim 10^{-9}$ (solar system)

Gauss - Bonnet Gravity

$$S = \int \sqrt{-g}\left[R^2 - 4R^{\mu\nu}R_{\mu\nu} + R^{\mu\nu\sigma\rho}R_{\mu\nu\sigma\rho}\right] d^4x$$

4D generalization of 2D gauss-bonnet theorem

$$\int_M K \, dA = 2\pi \chi(M)$$ (for geodesics)

extrinsic "Gaussian" curvature (geometry)

Euler characteristic (topology)

Basis of some string theories b/c when the classical limit is taken, higher order terms of curvature tensor appear. The Gauss-Bonnet term happens to be the unique quadratic curvature term to yield 2nd order equations

Brans - Dicke Gravity

- Still metric theory (like GR) but now scalar tensor theory.
- Fundamental variables now $g_{\mu\nu}$ and Φ.
- Φ has interpretation of changing gravitational constant in spacetime
- allows for additional vacuum solutions that GR doesn't

$$S = \frac{1}{16\pi}\int \sqrt{-g}\left[\Phi R - \frac{\omega}{\Phi}\partial_\mu \Phi \partial^\mu \Phi\right] d^4x + S_M$$

yields eqns of motion

$$\Box \phi = \frac{8\pi}{3+2\omega} T$$

energy momentum tensor for scalar fields

$$G_{\mu\nu} = \frac{8\pi}{\phi} T_{\mu\nu} + \frac{\omega}{\phi^2}\underbrace{\left(\partial_\mu \phi \partial_\nu \phi - \frac{1}{2} g_{\mu\nu} \partial_\lambda \phi \partial^\lambda \phi\right)}_{S_{\mu\nu}}$$

$$+ \frac{1}{\phi}\left(\nabla_\mu \nabla_\nu \phi - g_{\mu\nu} \Box \phi\right)$$

ϕ - scalar field
ω - dimensionless Dicke coupling constant
truly dimensionless

Throughout history ω has been constrained to be large. Currently $\sim 40,000$ by Cassini telescope. General relativity is limit as $\omega \to \infty$.

Can be seen by $\Box\phi \sim \mathcal{O}(\frac{1}{\omega})$
$\Rightarrow \phi \sim \phi_0 + \mathcal{O}(\frac{1}{\omega})$
constant, related to G

so all terms w/ derivative of ϕ drop out

One of the leading beyond GR theories

Einstein-Cartan
non-metric b/c there is torsion

$$T^\lambda{}_{\mu\nu} = \Gamma^\lambda{}_{\mu\nu} - \Gamma^\lambda{}_{\nu\mu} = 2\Gamma^\lambda_{[\mu\nu]} \neq 0$$

Still E-H action, still varying WRT the metric, but now a general metric connection is used (not christoffel connection)

Still yields $R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 8\pi T_{\mu\nu}^{SE}$
except $R_{[\mu\nu]} \neq 0$ so $T_{[\mu\nu]}^{SE} \neq 0$
can relate $T_{\mu\nu}^{SE}$ to spin tensor b/c of anti-symm part
in addition $T_{\mu\nu}{}^{\rho} + g_{\mu}{}^{\rho} T_{\nu\lambda}{}^{\lambda} - g_{\nu}{}^{\rho} T_{\mu\lambda}{}^{\lambda} = 4\pi \sigma_{\mu\nu}{}^{\rho}$

$\sigma_{\mu\nu}{}^{\rho}$ is spin tensor (like pauli matrices σ_i for spacetime)

- this only applies inside matter, in vacuum the torsion is 0.
- can be used to avoid singularities in your theorem b/c torsion. Thus singularities like inside BH and Big Bang are now "bounces" as interaction btwn torsion & spin prevent collapse (sort of like fermions)