# Dynamical Tides in Stars

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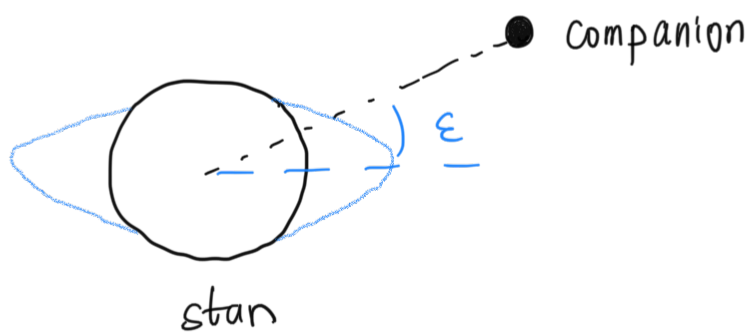
## – Equilibrium Tides vs Dynamical Tides, Tidal Q

Equilibrium tides: hydrostatic deformation of a star




Quality factor (Goldreich 1966)

$$\frac{1}{Q} = \frac{1}{2\pi E_0} \oint \left(-\frac{dE}{dt}\right) dt = \tan 2\varepsilon \quad \text{(MacDonald 1964)}$$

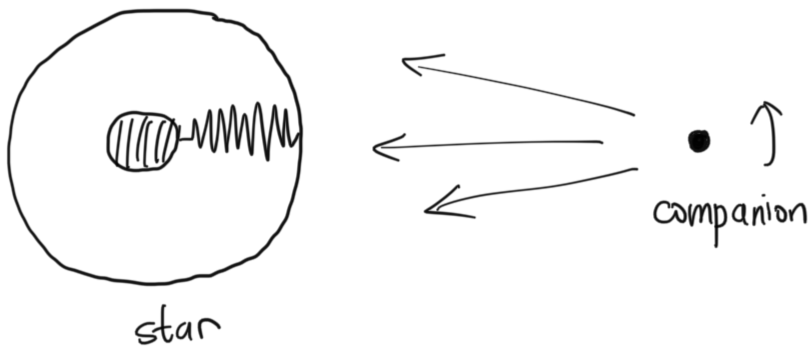
energy stored in tides ($E_0$)

dissipation in a complete cycle ($\oint (-\frac{dE}{dt}) dt$)

$\Rightarrow$ larger dissipation, smaller Q

## Dynamical tides : tidally excited oscillations (TEOs)

time variable nature of tidal forcing $\rightarrow$ excites oscillations




could be g-modes, p-modes, etc. (ask Nicholas)

Though Q is only physically defined for equilibrium tides, observers use them to characterize general tidal migration, even if it might be caused by dynamical tides.

### – Solving TEOs

We are trying to solve TEOs, i.e. which modes are excited under the tidal force, and how their amplitudes respond to it. We start with a non-rotating star.

Assume the background star is static, under linear theory, we have the perturbed equation

$$\delta\rho + \nabla\cdot(\rho\vec{\xi}) = 0 \qquad \text{(Continuity)}$$

$$\frac{\Delta p}{p} = \gamma\frac{\Delta\rho}{\rho} \qquad \text{(Energy)}$$

$$\frac{\partial^2\vec{\xi}}{\partial t^2} = -\frac{\nabla\delta p}{\rho} + \frac{\nabla p}{\rho^2}\delta\rho - \nabla\delta\phi - \nabla U \qquad \text{(Momentum)}$$

tidal potential

$$\nabla^2 \delta\phi = 4\pi G \delta\rho \qquad \text{(Gravity)}$$

where the Lagrangian perturbation variables $\Delta Q$ is related to the Eulerian perturbation variables $\delta Q$ by

$$\Delta Q = \delta Q + (\vec{\xi} \cdot \nabla) Q$$

Since these equations are all linear, it's possible to combine them to one equation of a single variable $\vec{\xi}$

$$\Rightarrow \quad \frac{\partial^2 \vec{\xi}}{\partial t^2} + C \cdot \vec{\xi} = -\nabla U \qquad (1)$$

where $C$ is a linear operator.

FYI,

$$C \cdot \vec{\xi} = \frac{1}{\rho}\Big[-\nabla(\gamma p \nabla \cdot \vec{\xi}) + \nabla p \nabla \cdot \vec{\xi} - (\nabla \vec{\xi}) \cdot (\nabla p) + \rho (\vec{\xi} \cdot \nabla) \nabla \phi + \rho \nabla \delta\phi \Big] \quad \text{(Schenk et al. 2002)}$$

To solve equation 1, we note that without the tidal potential, if we assume the solutions obey $\vec{\xi}_\alpha \propto e^{-i\omega_\alpha t}$, it reduces to the normal eigenvalue problem

$$C \cdot \vec{\xi}_\alpha = \omega_\alpha^2 \vec{\xi}_\alpha$$

We define an inner product $\langle \vec{\xi}_\alpha, \vec{\xi}_\beta \rangle = \int dV \rho \, \vec{\xi}_\alpha^* \vec{\xi}_\beta$, one can show that $C$ is a Hermitian operator, such that there exists a set of orthonormal solutions $\{\vec{\xi}_\alpha\}$, satisfying

$$\langle \vec{\xi}_\alpha, \vec{\xi}_\beta \rangle = \delta_{\alpha\beta}$$

So we can decompose the solution to equation 1, $\xi$, to

$$\vec{\xi} = \sum_\alpha a_\alpha(t)\vec{\xi}_\alpha \quad \text{where } a_\alpha(t) = \langle \vec{\xi}_\alpha, \vec{\xi}(t) \rangle,$$

then take "$\langle \vec{\xi}_\alpha |$" to Equation 1, we have

$$\Rightarrow \quad \langle \vec{\xi}_\alpha, \frac{\partial^2}{\partial t^2}\vec{\xi} \rangle + \langle \vec{\xi}_\alpha, C\vec{\xi} \rangle = -\langle \vec{\xi}_\alpha, \nabla U \rangle$$

$$\Rightarrow \ddot{a}_\alpha + \omega_\alpha^2 a_\alpha = -\langle \vec{\xi}_\alpha, \nabla U \rangle$$

So that one solves the TEOs for a non-rotating star. This is just a forced oscillator equation.

For a rotating star at spin $\vec{\Omega}_s$, equation 1 has one more term of the Coriolis force

$$\ddot{\vec{\xi}} + B\cdot\dot{\vec{\xi}} + C\cdot\vec{\xi} = -\nabla U \qquad (2)$$

where $B\cdot\dot{\vec{\xi}} = 2\vec{\Omega}_s \times \dot{\vec{\xi}}$

(Technically, the operator $C$ has one more term of the centrifigural force, but $C$ will still be Hermitian with that term, so it makes no difference for the formal analysis)

You may think that, let's do the trick again: find a set of $\{\vec{\xi}_\alpha\}$ satisfying the "free" equation

$$\mathcal{L}\vec{\xi} \equiv (-i\omega_\alpha B + C)\vec{\xi}_\alpha = \omega_\alpha^2 \vec{\xi}_\alpha$$

While one can show that $\mathcal{L}$ is Hermitian, the problem is, $\mathcal{L}$ has an explicit dependence on $\omega_\alpha$ ($\mathcal{L} = \mathcal{L}(\omega_\alpha)$), hence for different eigenvalue $\omega_\alpha$ and $\omega_\beta$, $\mathcal{L}_\alpha$ and $\mathcal{L}_\beta$ are NOT the same operator! So $\vec{\xi}_\alpha$ and $\vec{\xi}_\beta$ are not orthogonal.

If we try to find a complete set $\{\vec{\xi}_\alpha\}$ to solve for $\vec{\xi}$, the coefficients $a_\alpha$ are generally coupled to each other.

To tackle this problem, Dyson & Schutz 1979 points out that, instead of trying to expand $\vec{\xi}$ in the configuration space, one should use a phase space expansion. This means we should use the Hamiltonian equation of motion.

The Lagrangian to generate the equation of motion is

$$\mathcal{L} = \frac{1}{2}\dot{\vec{\xi}}\cdot\dot{\vec{\xi}} + \frac{1}{2}\dot{\vec{\xi}}\cdot B\cdot\vec{\xi} - \frac{1}{2}\vec{\xi}\cdot C\cdot\vec{\xi} - \nabla U\cdot\vec{\xi}$$

The canonical momentum is hence

$$\vec{\pi} = \frac{\partial \mathcal{L}}{\partial \dot{\vec{\xi}}} = \dot{\vec{\xi}} + \frac{1}{2}B\cdot\vec{\xi}$$

The Hamiltonian

$$\Rightarrow H = \dot{\vec{\xi}}\cdot\vec{\pi} - \mathcal{L} = \frac{1}{2}\left(\vec{\pi} - \frac{1}{2}B\cdot\vec{\xi}\right)^2 + \frac{1}{2}\vec{\xi}\cdot C\cdot\vec{\xi} + \nabla U\vec{\xi}$$

The Hamiltonian equation of motion

$$\begin{bmatrix} \dot{\vec{\xi}} \\ \dot{\vec{\pi}} \end{bmatrix} = \begin{bmatrix} -\frac{B}{2} & 1 \\ -C+\frac{1}{4}B^2 & -\frac{B}{2} \end{bmatrix} \begin{bmatrix} \vec{\xi} \\ \pi \end{bmatrix} + \begin{bmatrix} 0 \\ -\nabla U \end{bmatrix}$$

$$\text{or} \qquad \dot{\vec{\zeta}} = T\cdot\vec{\zeta} + \vec{F} \qquad (3)$$

We notice that then the "free" equation (without $\vec{F}$) becomes the normal eigenvalue problem

$$(T + i\omega_\alpha)\cdot\vec{\zeta}_\alpha = 0$$

However, since B is not Hermitian, T is not Hermitian. So $\{\vec{Z}_\alpha\}$ will generally not be a complete basis. However, one can obtain a complete basis by including all Jordan chains. Assume

$$V_A = \{\vec{Z}_\alpha | \vec{Z}_\alpha \text{ satisfying } (T + i\omega_\alpha)^\sigma \cdot \vec{Z}_\alpha = 0\}$$

for some $\sigma \geqslant 1$. Clearly $\vec{\zeta}_\alpha \in V_\alpha$. It can be shown that The direct sum of all $V_A$ gives the entire space. Then we can construct a set of $\vec{\zeta}_{\alpha,\sigma}$, $1 \leq \sigma \leq p_\alpha$, such that

$$(T + i\omega_\alpha) \cdot \vec{\zeta}_{\alpha,\sigma} = \vec{Z}_{\alpha,\sigma-1} \quad (\zeta_{\alpha,-1} = 0)$$

such that the set

$$\{\vec{\zeta}_{\alpha,\sigma} | \text{ all } \alpha s, 0 \leq \sigma \leq p_\alpha\}$$

forms a complete basis. But this basis is not necessarily orthogonal. But it is possible to choose a left basis $\{\vec{X}_{\alpha,\sigma}\}$ such that

$$(T^+ - i\omega_\alpha^*) \cdot \vec{X}_{\alpha,\sigma} = \vec{X}_{\alpha,\alpha-1}$$

and $\quad \langle \vec{X}_{\alpha,\sigma}, \vec{\zeta}_{\beta,\lambda} \rangle = \delta_{\alpha\beta} \delta_{\sigma+\lambda, p_\alpha}$

Let's assume we already have $\{\vec{\zeta}_{\alpha,\sigma}\}$ such that

$$\vec{\zeta} = \sum_{a,\sigma} a_{\alpha,\sigma}(t) \vec{\zeta}_{\alpha,\sigma}$$

Then equation 3 becomes

$$\sum_{\alpha,\sigma} \dot{a}_{\alpha,\sigma} \vec{Z}_{\alpha,\sigma} = \sum_{\alpha,\sigma} a_{\alpha,\sigma} T \vec{Z}_{\alpha,\sigma} + \vec{F}$$

$$= \sum_{\alpha,\sigma} a_{\alpha,\sigma} (-i\omega_\alpha \vec{Z}_{\alpha,\sigma} + \vec{Z}_{\alpha,\sigma-1}) + \vec{F}$$

with $\langle X_\beta, p_\beta - \lambda, \cdots \rangle$

$$\Rightarrow \quad \dot{a}_{\alpha,\sigma} + i\omega_\alpha a_{\alpha,\sigma} - a_{\alpha,\sigma+1} = \langle \vec{\chi}_{\alpha, p_\alpha - \sigma}, \vec{F} \rangle$$

where $a_{\alpha, p_A+1} = 0$

The ~~the~~ coefficients are coupled across $0 \le \sigma \le p_A$, but are not coupled across different $\alpha$s (different eigenvalues)

We already talked too much about linear algebra! From now on let's assume all Jordan chains have $p_A = 0$. A more general treatment can be seen in Schenk et al. 2002.

$$\Rightarrow \quad \dot{a}_\alpha + i\omega_\alpha a_\alpha = \langle \vec{\chi}_\alpha, \vec{F} \rangle \quad (4)$$

Next step: what is $\vec{\chi}_\alpha$? We know that $\vec{\zeta}_\alpha = \begin{bmatrix} \vec{\xi}_\alpha \\ \vec{\pi}_\alpha \end{bmatrix}$, assume $\vec{\chi}_\alpha = (\vec{\sigma}_\alpha, \vec{\tau}_\alpha)$. With $T^\dagger \vec{\chi}_\alpha = i\omega_\alpha^* \vec{\chi}_\alpha$, we must have $\vec{\sigma}_\alpha = i\omega_\alpha^* \vec{\tau}_\alpha - \frac{1}{2} B \cdot \vec{\tau}_\alpha$

We define a symplectic matrix $M = \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix}$, satisfying

$$M^\dagger = M = M^{-1}, \quad M^\dagger T M = -T^\dagger$$

If $(T + i\omega_\alpha)\vec{\zeta}_\alpha = 0$, then $(T^\dagger - i\omega_\alpha^*) M \vec{\zeta}_\alpha = 0$

Since $M$ is invertable, $M\vec{\zeta}_\alpha$ must form a basis. If there's no degeneracy, we must have

$$\vec{\chi}_\alpha = b_\alpha M \vec{\zeta}_\alpha \Rightarrow \vec{\tau}_\alpha = -i b_\alpha \vec{\xi}_\alpha$$

The orthonormal relation requires

$$\delta_{\alpha\beta} = \langle \vec{\chi}_\alpha | \vec{\zeta}_\beta \rangle = (i\omega_\alpha^* \tau_\alpha - \tfrac{1}{2} B \vec{\tau}_\alpha, \vec{\tau}_\alpha) \begin{pmatrix} \vec{\xi}_\beta \\ -i\omega_\beta \vec{\xi}_\beta + \frac{1}{2} B \cdot \vec{\xi}_\beta \end{pmatrix}$$

$$= -i(\omega_\alpha + \omega_\beta) \langle \vec{\tau}_\alpha, \vec{\xi}_\beta \rangle + \langle \vec{\tau}_\alpha, B \cdot \vec{\xi}_\beta \rangle$$

$$= (\omega_\alpha + \omega_\beta) b_\alpha^* \langle \vec{\xi}_\alpha, \vec{\xi}_\beta \rangle + i b_\alpha^* \langle \vec{\xi}_\alpha, B \vec{\xi}_\beta \rangle$$

when $\alpha = \beta$, $\langle \vec{\xi}_\alpha, \vec{\xi}_\alpha \rangle = 1$, $\Rightarrow 1 = 2 \varepsilon_\alpha b_\alpha^*$

where $\varepsilon_\alpha = \omega_\alpha + \frac{i}{2} \langle \vec{\xi}_\alpha, B \vec{\xi}_\beta \rangle \approx \omega_\alpha$

$$\Rightarrow b_\alpha = \frac{1}{2\varepsilon_\alpha^*}, \text{ or } \quad \vec{\tau}_\alpha = -\frac{i}{2\varepsilon_\alpha^*} \vec{\xi}_\alpha$$

$$\Rightarrow \langle \vec{\xi}_\alpha, \vec{\xi}_\beta \rangle = \delta_{\alpha\beta} - \frac{i}{\omega_\alpha + \omega_\beta} \langle \vec{\xi}_\alpha, B \vec{\xi}_\beta \rangle$$

$$= \underbrace{\delta_{\alpha\beta} - \frac{2}{\omega_\alpha + \omega_\beta} \int dV \rho \, \vec{\xi}_\alpha^* \cdot (i \vec{\Omega}_s \times \vec{\xi}_\beta)}_{\text{modified orthonormality relation}}$$

Back to (4)

$$\dot{a}_\alpha + i \omega_\alpha a_\alpha = \langle \vec{\chi}_\alpha, \vec{F} \rangle = (\vec{\sigma}_\alpha, \vec{\tau}_\alpha) \begin{pmatrix} 0 \\ -\nabla U \end{pmatrix}$$

$$= \langle \vec{\tau}_\alpha, -\nabla U \rangle = \frac{i}{2\varepsilon_\alpha} \langle \vec{\xi}_\alpha, -\nabla U \rangle \quad (5)$$

For a more general discussion regarding degeneracy, Jordan chain, see Schenk et al. 2002

The Result:

$$\begin{cases} \vec{\xi} = \sum_\alpha a_\alpha(t) \vec{\xi}_\alpha \\ \dot{a}_\alpha + i \omega_\alpha a_\alpha = \frac{i}{2\omega_\alpha} \langle \vec{\xi}_\alpha, -\nabla U \rangle \end{cases}$$

We can solve this directly, but let's make some approximations

We can seperate $\vec{\xi} = \vec{\xi}_{dyn} + \vec{\xi}_{eq}$, where $\vec{\xi}_{eq}$ defined by the static limit of (2)

$$C \cdot \vec{\xi}_{eq} \equiv -\nabla U$$

This implies if $U \propto e^{-i\omega_f t}$, where $\omega_f$ is the tidal forcing frequency, $\vec{\xi}_{eq} \propto e^{-i\omega t}$, so $\sum \dot{a}_{\alpha,eq} \vec{\xi}_\alpha = -i\omega \sum a_{\alpha,eq} \vec{\xi}_\alpha$

It is then possible to extract $\vec{\xi}_{eq}$ from $\vec{\xi}$, and we can get an equation of $a_{\alpha,dyn}$ (Fuller 2017)

$$\dot{a}_{\alpha,dyn} + i\omega_\alpha a_{\alpha,dyn} = \underbrace{\frac{i\omega_f}{2\omega_\alpha^2} \langle \vec{\xi}_\alpha, -\nabla U \rangle}_{\text{driving term by tidal forcing}} + \underbrace{\frac{i\omega_f}{2\omega_\alpha} (\omega_f - \omega_\alpha) \langle \vec{\xi}_\alpha, \vec{\xi}_{eq} \rangle}_{\text{damping on the equilibrium tide}}$$

With the adiabatic approximation, $a_{\alpha,dyn} \propto e^{-i\omega t}$

$$a_{\alpha,dyn} = \underbrace{\frac{1}{2} \frac{\omega_f}{\omega_\alpha - \omega_f} \frac{\langle \vec{\xi}_\alpha, -\nabla U \rangle}{\omega_\alpha^2}}_{\substack{\text{resonance term} \\ \text{large when } \omega_f \sim \omega_\alpha}} - \underbrace{\frac{1}{2} \frac{\omega_f}{\omega_\alpha} \langle \vec{\xi}_\alpha, \vec{\xi}_{eq} \rangle}_{\text{ignorable}}$$

$$\Rightarrow \quad a_{\alpha,dyn} = \frac{1}{2} \frac{\omega_f}{\omega_\alpha - \omega_f + i\gamma_\alpha} \frac{\langle \vec{\xi}_\alpha, -\nabla U \rangle}{\omega_\alpha^2}$$

What is $U$? For aligned spin and orbit (Fuller & Lai 2012)

$$U(\vec{r}, t) = -GM_2 \sum_{lm} \frac{W_{lm} r^l}{D^{l+1}} e^{-im\Phi} Y_{lm}(\theta, \phi + \Omega_s t)$$

$D$ is the binary separation, $\Phi$ is the orbital true anomaly

$W_{lm}$ are some coefficients. $W_{20} = -\sqrt{\frac{\pi}{5}}$, $W_{22,-2} = \sqrt{\frac{3\pi}{10}}$

It turns out one can derive the energy and angular momentum of each mode $\vec{\xi}_\alpha$, from the Hamiltonian (density)

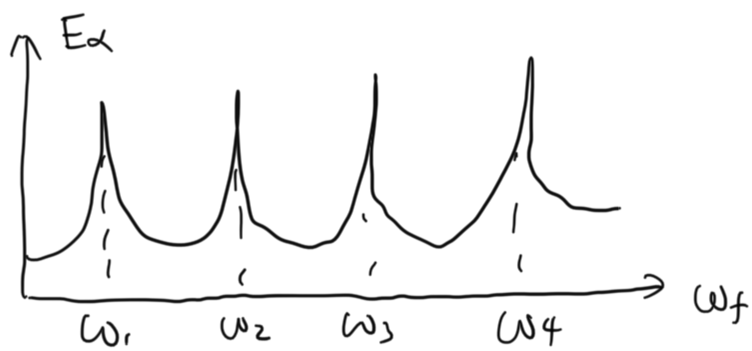
$$H = \underbrace{\frac{1}{2}\dot{\xi}\cdot\dot{\xi}}_{E_k} + \underbrace{\frac{1}{2}\xi\cdot C\cdot\xi}_{E_p} + \underbrace{\nabla U\cdot\xi}_{E_{\text{coupling}}}$$

$E_{\text{coupling}}$ is not going to dissipate. $E_\alpha = E_k + E_p \sim 2E_k$ by equipartition

$$\Rightarrow \quad E_\alpha \propto |a_{\alpha,\text{dyn}}|^2 \propto \frac{\omega_f^2}{(\omega_\alpha-\omega_f)^2+\gamma_\alpha^2} W_{em}^2$$

$$J_\alpha \propto m|a_{\alpha,\text{dyn}}|^2 \propto m E_\alpha$$

$\Rightarrow$ There is a resonance structure of $E_\alpha$




- Applications

i) Can we observe TEOs? Yes!

Heartbeat Stars (Fuller 2017)

ii) Circulization of Orbits for $m=0$ modes

Since $J_\alpha \propto m$. TEOs with $m=0$ do not dissipate angular momentum, but dissipates energy. This helps to circularize the orbits. (e.g. Zanazzi & Wu 2021)

iii) Resonance Locking (e.g. Ma & Fuller 2021)


$\Rightarrow t_{\text{tidal migration}} \sim t_{\text{modes}}$

$\Rightarrow$ different $Q$s, contract to equilibrium tides

e.g. Saturn moons (Lainey et al 2019
Fuller et al 2016)