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ARC Journal Club
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Introduction to Asteroseismology
18 April 2022

Outline *These Notes are an introduction to some important concepts within asteroseismology. We first derive the dispersion relation for oscillation modes under the influence of pressure and buoyancy, identifying and characterizing p mode and g modes. We then discuss some observational properties of stellar oscillation spectra, and translate them to physical properties of a star. Finally, we briefly describe the impact of rotation and magnetism on oscillation spectra.*

Contents

1 Dispersion relation for stellar oscillations

In this section, we derive a rough radial dispersion relation for non-radial oscillations in perfectly spherical stars under only the influence of pressure and buoyancy. In particular, this means that we neglect effects such as rotation, magnetic fields, and shear restorative forces, all of which may be important in some situations but which dramatically complicate the mathematics.

The derivation shown here can be found in standard texts such as [1].

1.1 Full fluid equations

There are five fluid equations which we should solve simultaneously, reflecting the five fluid properties that we are interested in keeping track of (the density ρ, pressure p, and three components of the fluid velocity $\vec{u}$). We will write these equations in terms of the convective derivative (the **Lagrangian** perspective), tracking the properties of a fluid parcel which we follow as it moves around, i.e.,

$$\frac{\mathrm{d}f}{\mathrm{d}t} \equiv \frac{\partial f}{\partial t} + (\vec{u} \cdot \nabla) f \tag{1}$$

First, the **continuity equation** is

$$\frac{\mathrm{d}\rho}{\mathrm{d}t} + \rho \nabla \cdot \vec{u} = 0 \tag{2}$$

The three components of the **momentum equation** can be written in the form

$$\rho \frac{\mathrm{d}\vec{u}}{\mathrm{d}t} = -\nabla p - \rho g \hat{r} \tag{3}$$

where $g = g(r)$ is the gravitational acceleration at radius r. Finally, to close the equations, we will write down the **energy equation**, which relates the density of a fluid parcel to its pressure:

$$\frac{\mathrm{d}\ln p}{\mathrm{d}t} = \gamma \frac{\mathrm{d}\ln \rho}{\mathrm{d}t} \tag{4}$$

This reflects the **adiabatic** assumption that a given fluid parcel's pressure is related to its density by $p \propto \rho^\gamma$. It is important to note that this relation is *only* true for a given fluid parcel, and the prefactor of this relation depends on the specific entropy of the fluid parcel. In other words, it is very important that we write down Equation 4 with convective, not partial, derivatives.

1.2 Linearized equations

In order to find the oscillation modes of the star, we assume that all of the perturbations are proportional to $e^{i\omega t}$. In practice, this means that we can replace time derivatives $\partial/\partial t$ by $i\omega$. Note that the fluid displacements are related to the fluid velocities by $\vec{u} = \mathrm{d}\vec{\xi}/\mathrm{d}t$.

We will also **linearize** the equations by assuming that all perturbations can be decomposed into a large, time-independent equilibrium part and a small perturbation. We can write

$$\begin{aligned} \rho(\vec{x}, t) &= \rho_0(r) + \rho'(\vec{x}, t) \\ p(\vec{x}, t) &= p_0(r) + p'(\vec{x}, t) \end{aligned} \tag{5}$$

Note that the equilibrium fluid displacement/velocity is zero, so $\vec{u}$ (or $\vec{\xi}$) is already a small quantity. Note that, under this assumption, the convective derivative of a perturbation is the same as its partial time derivative, e.g., $\mathrm{d}\rho'/\mathrm{d}t \approx \partial\rho/\partial t$.

Then, keeping only terms first-order in the perturbations and working in spherical coordinates, we have

$$\boxed{\frac{\rho'}{\rho_0} + \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\xi_r\right) + \nabla_h \cdot \vec{\xi}_h = 0} \quad \text{(continuity)} \tag{6}$$

where we have defined $\vec{\xi}_h = \vec{\xi} - \xi_r\hat{r}$ to be the horizontal fluid displacement, and $\nabla_h = \nabla - (\partial/\partial r)\hat{r}$ to be the horizontal derivative operator.

The momentum equation becomes

$$\boxed{-\rho_0\omega^2\xi_r = -\frac{\partial p'}{\partial r} - \rho' g} \quad \text{(radial momentum)} \tag{7}$$

$$\boxed{-\rho_0\omega^2\vec{\xi}_h = -\nabla_h p'} \quad \text{(horizontal momentum)} \tag{8}$$

We have made the **Cowling approximation**, that the gravity g remains its equilibrium quantity. Not making this approximation entails writing and solving Poisson's equation, which complicates the mathematics.

For the energy equation, we can expand the convective derivatives, keep only linear terms, and rearrange:

$$\frac{\rho'}{\rho_0} - \frac{p'}{\gamma p_0} = \xi_r\left(\frac{1}{\gamma}\frac{\mathrm{d}\ln p_0}{\mathrm{d}r} - \frac{\mathrm{d}\ln\rho_0}{\mathrm{d}r}\right) \tag{9}$$

Define the **Brunt–Väisälä frequency** (or buoyancy frequency) N as

$$N^2 \equiv g\left(\frac{1}{\gamma}\frac{\mathrm{d}\ln p_0}{\mathrm{d}r} - \frac{\mathrm{d}\ln\rho_0}{\mathrm{d}r}\right) \tag{10}$$

Note that N^2 can be either positive or negative. When $N^2 > 0$, N corresponds to the frequency of oscillation of a radially displaced fluid parcel in a radiative (i.e., convectively stable, stably stratified) zone. When $N^2 < 0$, the zone is convectively unstable and cannot support gravity waves.

We also define the **sound speed** as

$$c_s^2 \equiv \frac{\gamma p_0}{\rho_0} \tag{11}$$

The energy equation then becomes

$$\boxed{p' = \rho' c_s^2 - \frac{\rho_0 N^2 c_s^2}{g}\xi_r} \quad \text{(energy)} \tag{12}$$

Intuitively, this form of the energy equation reflects the two different ways that an Eulerian pressure perturbation can change. The first term on the right hand side simply relates p' to the density perturbation via the sound speed. The second term represents the fact that, when there is some nonzero radial displacement ξ_r, the fluid parcel which now occupies a given position has a slightly different *equilibrium* pressure (encoded by N^2/g).

1.3 Horizontal eigenfunctions

Before we try to solve the equations, it would first be beneficial to derive the eigenfunctions of the horizontal Laplacian operator ∇_h^2, whose action on a function $Y = Y(\theta, \phi)$ is

$$\nabla_h^2 Y(\theta,\phi) = \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial Y(\theta,\phi)}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 Y(\theta,\phi)}{\partial\phi^2} \tag{13}$$

Exploiting the axisymmetry of the problem, we can take $Y(\theta,\phi) = f(\theta)e^{im\phi}$ for an integer m. Then

$$\nabla_h^2 f(\theta) = \frac{1}{r^2\sin\theta}\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\sin\theta\frac{\mathrm{d}f(\theta)}{\mathrm{d}\theta}\right) - \frac{m^2}{r^2\sin^2\theta}f(\theta) \tag{14}$$

Now take $\mu = \cos\theta$. Then the eigenvalue equation for ∇_h^2 for eigenvalue $-\lambda/r^2$ becomes

$$\frac{\mathrm{d}}{\mathrm{d}\mu}\left(\left(1-\mu^2\right)\frac{\mathrm{d}f(\mu)}{\mathrm{d}\mu}\right) - \frac{m^2}{1-\mu^2}f(\mu) = -\lambda f(\mu) \tag{15}$$

Equation 15 is the **general Legendre equation**, and is solved by the **associated Legendre polynomials** $f(\mu) = P_\ell^m(\mu)$ with eigenvalue $\lambda = \ell(\ell+1)$ such that $\ell \geq m$ is a non-negative integer.

We then see that the eigenfunctions are the **spherical harmonics** $Y_\ell^m(\theta,\phi)$, defined to be

$$Y(\theta,\phi) = P_\ell^m(\cos\theta)e^{im\phi} \tag{16}$$

1.4 Solving the linearized equations

The broad goal in solving the linearized equations will be to write all the equations in terms of the pressure (p') and radial displacement (ξ_r) perturbations. First, we take the horizontal divergence of the horizontal momentum equation (Equation 8) to find

$$\nabla_h \cdot \vec{\xi}_h = \frac{1}{\rho_0\omega^2}\nabla_h^2 p' \tag{17}$$

The energy equation (Equation 12) can be solved for ρ' to obtain

$$\rho' = \frac{1}{c_s^2}p' + \frac{\rho_0 N^2}{g}\xi_r \tag{18}$$

We can then substitute Equations 17 and 18 into the linearized continuity equation (Equation 6) to obtain

$$\frac{1}{\rho_0 c_s^2}p' + \frac{N^2}{g}\xi_r + \frac{\partial\xi_r}{\partial r} + \frac{2}{r}\xi_r + \frac{1}{\rho_0\omega^2}\nabla_h^2 p' = 0 \tag{19}$$

Assuming that the horizontal dependences of all of the perturbations are spherical harmonics (Equation 16), Equation 19 becomes

$$\boxed{\frac{\mathrm{d}\xi_r}{\mathrm{d}r} = -\left(\frac{2}{r} + \frac{N^2}{g}\right)\xi_r + \frac{1}{\rho_0 c_s^2}\left(\frac{S_\ell^2}{\omega^2} - 1\right)p'} \tag{20}$$

Here, we have defined the **Lamb frequency** as

$$S_\ell^2 = \frac{\ell(\ell+1)c_s^2}{r^2} \tag{21}$$

The Lamb frequency can be interpreted as the frequency of an acoustic wave with horizontal wavenumber $k_h \equiv \sqrt{\ell(\ell+1)}/r$ via $S_\ell^2 = k_h^2 c_s^2$.

We can also substitute our Equation 18 into Equation 7 to obtain

$$\boxed{\frac{\mathrm{d}p'}{\mathrm{d}r} = \rho_0\left(\omega^2 - N^2\right)\xi_r - \frac{g}{c_s^2}p'} \tag{22}$$

Equations 20 and 22 give the radial derivatives of ξ_r and p' in terms of ξ_r and p'. As a very rough approximation, we can ignore the first term of Equation 19 and the second term of Equation 22—for high radial wavenumbers, we can argue that it is approximately fine to neglect terms $\sim 1/r$ or inverse scale heights of equilibrium quantities.

Then, solving for p' in Equation 20 and substituting into Equation 22, we obtain a crucial heuristic equation in asteroseismology:

$$\boxed{\frac{\mathrm{d}^2\xi_r}{\mathrm{d}r^2} = -\frac{1}{\omega^2 c_s^2}\left(\omega^2 - N^2\right)\left(\omega^2 - S_\ell^2\right)\xi_r} \tag{23}$$

If we roughly write $\xi_r \propto e^{i\int^r k_r(r')dr'}$ where $k_r \gg 1/r$ (the **Wentzel–Kramers–Brillouin approximation**, or WKB approximation), this becomes

$$\boxed{k_r^2 c_s^2 = \frac{1}{\omega^2}\left(\omega^2 - N^2\right)\left(\omega^2 - S_\ell^2\right)} \tag{24}$$

1.5 Acoustic modes

When $\omega^2 \gtrsim N^2, S_\ell^2$ (high frequency), the dispersion relation becomes

$$\omega^2 = k_r^2 c_s^2 \tag{25}$$

We recognize this as the acoustic dispersion relation for a sound wave (note that we have already assumed that $k_r \gg k_h$ by assuming that terms $\sim 1/r$ are small). This is a **pressure wave**, also called a **p mode**.

1.6 Gravity modes

When $\omega^2 \lesssim N^2, S_\ell^2$ (low frequency), the dispersion relation becomes

$$\omega^2 = \frac{k_h^2}{k_r^2} N^2 \tag{26}$$

We recognize this as the dispersion relation of **gravity waves**, or **g modes**, in the limit where $k_r \gg k_h$.

Note that g modes are impossible in convectively unstable regions, those where $N^2 < 0$. In these cases, there can only be p modes when $\omega^2 \gtrsim S_\ell^2$.

Note that, even though technically N^2 is nonzero in a convective zone, in practice convection is so efficient at mixing entropy superadiabatically that N^2 will be forced to be very small.

Also, note that $S_0^2 = 0$. Therefore, it is not possible for $\ell = 0$ (radial) oscillations to be gravity modes.

1.7 Evanescent regions

When ω^2 is in between N^2 and S_ℓ^2 (either $N^2 < \omega^2 < S_\ell^2$ or $S_\ell^2 < \omega^2 < N^2$), the dispersion relation will imply that

$$k_r^2 < 0 \tag{27}$$

In other words, waves in these regions are **evanescent**—they will exponentially grow or decay with radius, but will not be **propagating** like they would be in the p mode or g mode regions.

1.8 Propagation diagrams

A given oscillation mode has a single ω throughout the entire star, by definition. Therefore, the **propagation diagram** is an extremely useful tool in asteroseismology for visualizing the radial structure of a star and the behavior of that star's oscillation modes. One such diagram is shown in Figure 1.

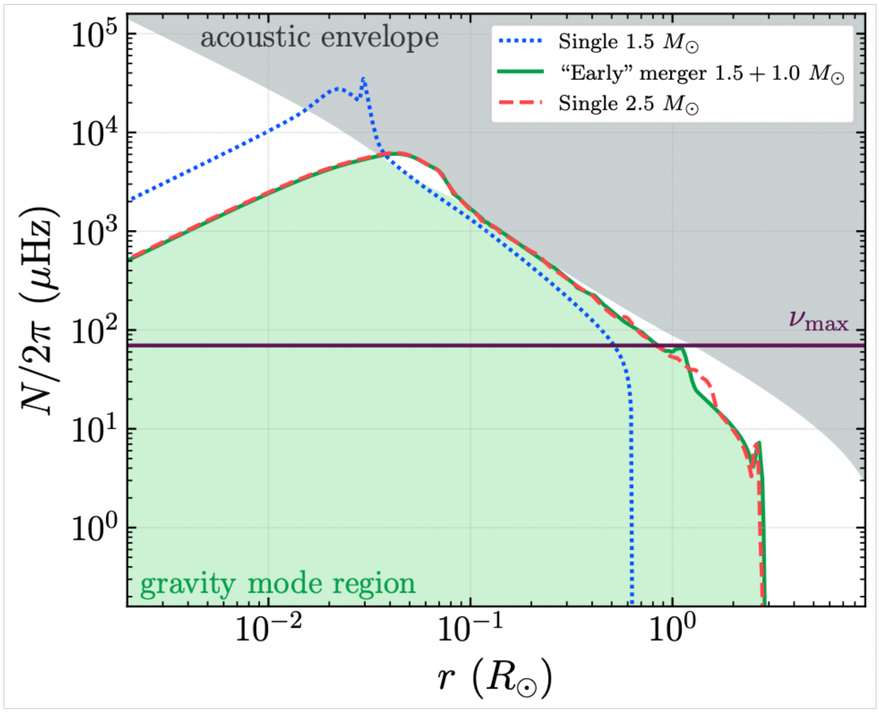

Figure 1: A propagation diagram showing the Brunt–Väisälä frequency profiles for three different stellar models, as well as the $\ell = 1$ Lamb frequency profile for one of them. The p mode region is shown in gray, the g mode region is shown in green, and the evanescent region is shown in green. The horizontal line labeled $\nu_{\rm max}$ is a representative oscillation frequency. This figure is reproduced from the left panel of Figure 4 in [2].

From Figure 1, a few interesting things can be observed. First, typically the Lamb frequency is larger (sometimes much larger) than the Brunt–Väisälä frequency. This is particularly true in main sequence stars.

We observe that many stars (e.g., red giants, the Sun, etc.) have surface convection zones, meaning that they can only maintain p modes on the surface. If there is also a radially large evanescent region, then it will be very hard to detect g modes, if it is even possible at all.

In some situations, like red giants (like the models shown in Figure 1), the most excited modes lie in regions where the evanescent zone has very small vertical extent. This means that the p mode oscillations at the surface of the star couple to the g mode oscillations at the center of the star, since there is not much amplitude decay within the evanescent region. Such modes are typically called **mixed modes**, and will have the character of both p modes and g modes. The coincidence that red giants are excited at such convenient frequencies means that g modes can be effectively leveraged to learn about their cores.

2 Measuring oscillation modes to extract physics

2.1 Mode quantization

A star is a three-dimensional problem. Therefore, we expect three quantization conditions. The first and second quantization condition came from solving for the horizontal eigenfunctions, giving ℓ and m as quantum numbers. The final quantization condition comes from imposing boundary conditions at two radii in the star (usually at the center and the surface), which quantizes the frequencies ω which modes can have. It is common to use the Eckart scheme for counting radial orders as $n = n_p - n_g$, where n_p and n_g are the number of zero-crossings in the p mode and g mode regions, respectively [3].

Software such as GYRE [4] can be used to find such modes. However, some heuristics exist for relating features of the mode spectrum to relate properties of a star's oscillation spectrum to the physical parameters of the star.

2.2 Observing stellar oscillations

Observationally, oscillation modes can be measured in two different ways, and such measurements always reflect the amplitude of the mode at or near the surface of the star:

- **Photometry:** They can be measured by taking the **power spectrum** of the **light curve** of the star, typically over a period of months or even years. The light curve mostly reflects changes in brightness due to temperature perturbations on the surface of the star.
- **Spectroscopy:** They can be measured by tracing the **radial velocities** of fluid at the surface of the star.

Because of effects like **geometric cancellation** (adjacent outwards and inwards-moving cells "cancelling" each other out in, e.g., brightness), it is extremely hard to measure modes with $\ell \gtrsim 3$, with the amplitude of modes decreasing as $\propto 1/\sqrt{\ell}$ (see, e.g., [5]).

Some examples of oscillation spectra are shown in Figure 2. Asteroseismic observations typically (but do not always) come from big observational surveys such as *Kepler*, *CoRoT*, and (more recently) *TESS*.

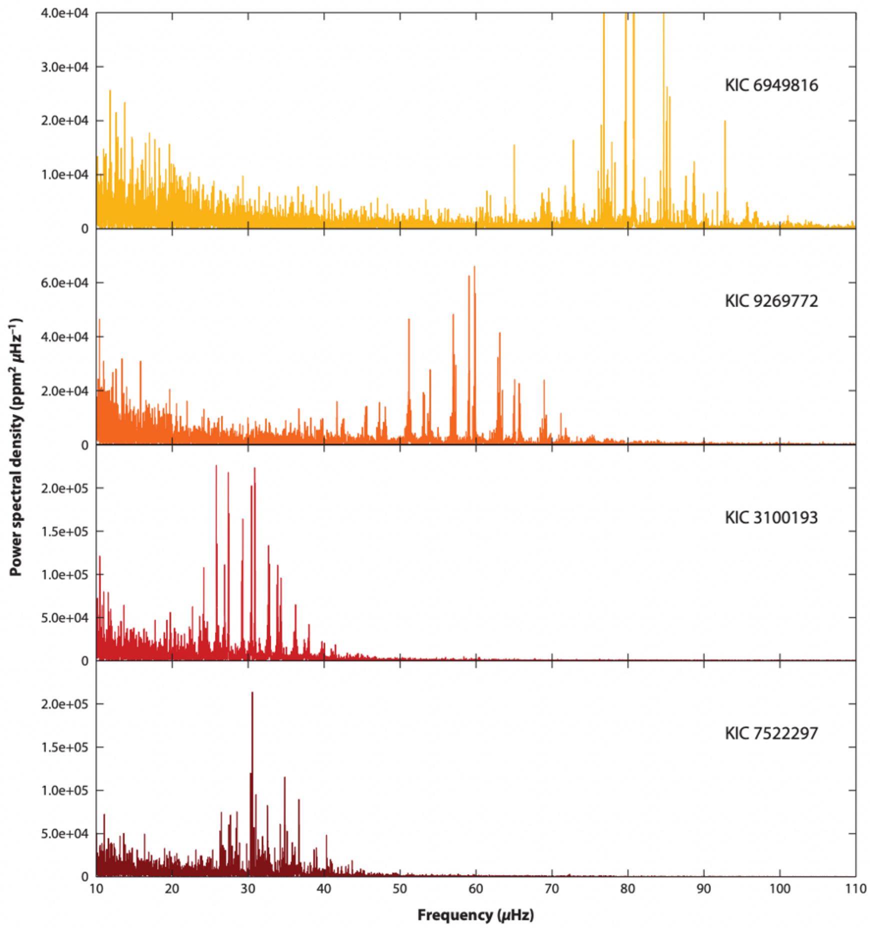

Figure 2: Four solar-like oscillation spectra measured by *Kepler*, reproduced from the Figure 3 of [6]. These four stars are all RGB or RC stars.

2.3 ν_{max} and $\Delta\nu$

In this section, I will primarily focus on red giant stars, because of their strong mixed-mode characters.

The power spectrum of a red giant consists of many discrete modes whose amplitudes follow a broad, peaked shape (for examples, see Figure 2). Even though an infinite number of modes exist all across the frequency spectum, not all of those modes will *necessarily* be driven, and driving must come from some physical source.

The rough frequency at which the amplitude of the modes is the highest is called the **frequency of maximum power** or ν_{max}. It is difficult to derive ν_{max} from first principles. However, it can be estimated reasonably effectively by arguing that the oscillations in a red giant will be primarily driven by convective motions in the envelope of the star. In particular, we can estimate that ν_{max} is the frequency of oscillation of a fluid blob moving through a pressure scale height $H_p \sim p/\rho g$ at the sound speed $c_s \sim p^{1/2}/\rho^{1/2}$. Then

$$\boxed{\nu_{\max} \sim \frac{c_s}{H_p} \sim \frac{g\rho^{1/2}}{p^{1/2}} \sim \frac{m^{1/2}g}{k^{1/2}T_{\text{eff}}^{1/2}} \propto MR^{-2}T_{\text{eff}}^{-1/2}} \tag{28}$$

where we have used the ideal gas law.

Adjacent modes of different radial order but the same ℓ are separated by the **large frequency spacing** $\Delta\nu$, which is roughly the sound-speed crossing time through the entire star. Using $c_s \sim p^{1/2}/\rho^{1/2} \sim \sqrt{GM/R}$ from the characteristic pressure and density scales of a star, we have

$$\boxed{\Delta\nu \sim \frac{c_s}{R} \sim \sqrt{G\bar{\rho}} \propto M^{1/2}R^{-3/2}} \tag{29}$$

where $\bar{\rho}$ is the average density of the star.

Together with a measurement of T_{eff} (usually from stellar photometry), measurements of $\nu_{\max}$ and $\Delta\nu$ can be used to extract the mass and radius of a star using Equations 28 and 29 (for some often Sun-based calibration).

2.4 ΔP_g

When mixed modes couple to the gravity wave-dominated core of a star (e.g., a red giant), a given acoustic mode will appear to split in the power spectrum into many modes which are closely spaced by roughly the **g mode period spacing** ΔP_g. Note that, unlike in the case with $\Delta\nu$, this splitting is roughly equal in *period*, not *frequency*.

Heuristically, one imagines that the "same" acoustic mode is coupling to many different gravity modes in the core, manifesting in many modes whose frequencies are all approximately (but not exactly) that of the acoustic mode.

The period spacing is roughly given by an integral of the Brunt–Väisälä frequency over the core of the star:

$$\boxed{\Delta P_g^{-1} = \frac{2\pi^2}{\sqrt{\ell(\ell+1)}} \int_{\nu_{\max}<N} \frac{N}{r}\,\mathrm{d}r} \tag{30}$$

3 Effect of other physics

So far, we have described the oscillation modes of a spherical star under pressure and buoyancy *only*. In this section, we explore the effects of other forces on stellar oscillation spectra.

3.1 Rotation

A standard reference for the content of this section is [7].

In addition to distorting the star itself, rotation contributes a **Coriolis force** ($\propto \Omega$) and **centrifugal force** ($\propto \Omega^2$). It is common to neglect the distortion of the star and the centrifugal force for analytical convenience, and only include the Coriolis force in the momentum equation (this is reasonable at low rotation rates).

Rotation breaks the spherical symmetry of the star by defining a special direction via

$$\vec{\Omega} = \Omega\hat{z} = \Omega\cos\theta\hat{r} - \Omega\sin\theta\hat{\theta} \tag{31}$$

However, azimuthal symmetry is maintained, and perturbations can still be assumed to be proportional to $e^{im\phi}$ for integer m (however, the sign of $m\Omega/\omega$ determines whether the mode is **prograde** or **retrograde**).

If the WKB approximation is assumed in both directions, it can be shown that the dispersion relation becomes (e.g., [8])

$$\boxed{\omega^2 - \frac{k_h^2}{k^2}N^2 - \left(\vec{k}\cdot\frac{2\vec{\Omega}}{k}\right)^2 = 0} \tag{32}$$

Note that, if $|N| \gg |\omega|, |\Omega|$ (which is typical of, e.g., gravity waves), then $k_h \ll k_r$. Therefore, $\vec{\Omega}\cdot\vec{k} \approx \Omega\cos\theta k_r$. The realization that the $\hat{\theta}$ component of $\vec{\Omega}$ is less important justifies dropping this component. This approximation is called the **traditional approximation of rotation** (**TAR**), and miraculously makes the radial and horizontal fluid equations separable.

Under the TAR, the θ dependence $\Theta = \Theta(\mu)$ are no longer associated Legendre polynomials, but are solutions to the **Laplace tidal equation**,

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}\mu}\left(\frac{1-\mu^2}{1-\mu^2\nu^2}\frac{\mathrm{d}\Theta(\mu)}{\mathrm{d}\mu}\right) - \frac{1}{1-\mu^2\nu^2}\left(\frac{m^2}{1-\mu^2} + m\nu\frac{1+\mu^2\nu^2}{1-\mu^2\nu^2}\right)\Theta(\mu) = -\lambda\Theta(\mu)} \tag{33}$$

where $\nu \equiv 2\Omega/\omega$ (note that this becomes the generalized Legendre equation when $\nu = 0$). The horizontal wavenumber then becomes $k_h \sim \sqrt{\lambda}/r$ (where $\lambda \neq \ell(\ell+1)$ in general).

The functions $\Theta(\mu)$ are called **Hough functions** and have some interesting properties.

- For large ν, the functions become $\lambda = (2l_\mu - 1)^2\nu^2$ (where l_μ is the number of nodes). A heuristic argument for this is given in [9], and a more detailed treatment can be found in [10, 11].
- For large ν, the $\lambda > 0$ modes become more and more confined to the equatorial region of the star (within $|\mu| \lesssim 1/\nu$). Roughly speaking, the combination $\mu\nu$ can be seen to "set the scale" of the function's dependence on μ.
- For $|\nu| > 1$, the coefficients of the equation change sign over the domain, and an infinite $\lambda < 0$ spectrum is present.
- If $\lambda < 0$, the sign of $\left(\omega^2 - k_h^2 c_s^2\right)\left(\omega^2 - N^2\right)$ can still be positive even if $N^2 < 0$ (convective region). Therefore, otherwise unstable gravity modes in the *convective* region of the star can be stabilized by the Coriolis force.

Figure 3 shows λ as a function of ν for $m = -2$.

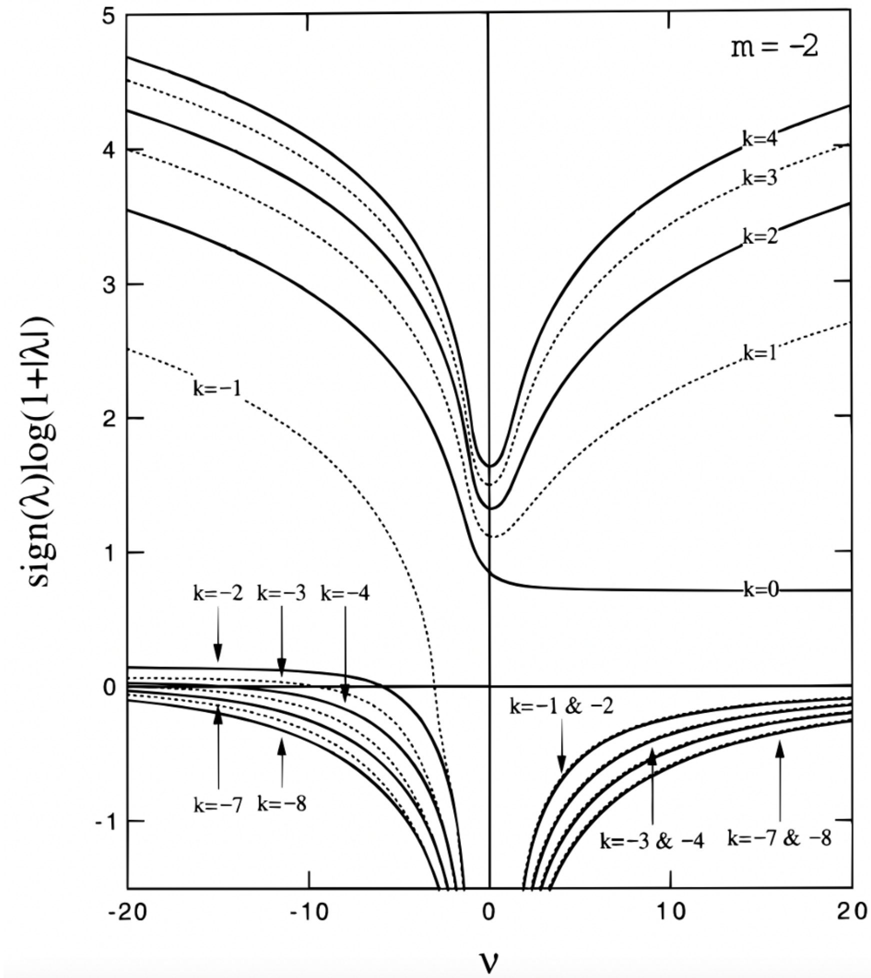

Figure 3: The eigenvalues of the Laplace tidal equation λ as a function of ν for $m = -2$, reproduced from Figure 1 of [7].

3.2 Magnetism

For a local WKB analysis and specializing to magneto-gravity waves only ($\omega \ll N, S_\ell$), the dispersion relation becomes (e.g., [8])

$$\boxed{\omega^2 - \frac{k_h^2}{k^2}N^2 - \left(\vec{k}\cdot\vec{v}_A\right)^2 = 0} \tag{34}$$

We note the similarity to Equation 32, except with the Alfven velocity $\vec{v}_A$ taking the place of $2\vec{\Omega}/k$. Note that $\vec{v}_A = \vec{B}_0/\sqrt{4\pi\rho_0}$ is complicatedly dependent on the magnetic field geometry, stellar profile, and $\vec{k}$.

Using $k_r \gg k_h$ so that $k \approx k_r$, we can write

$$k_r^4 v_{A,r}^2 - k_r^2 \omega^2 + k_h^2 N^2 = 0 \tag{35}$$

This is a quadratic in k_r^2, and is solved by (e.g., [12])

$$k_r^2 = \frac{\omega^2}{2v_{A,r}^2} \left(1 \pm \sqrt{1 - \frac{4v_{A,r}^2 N^2 k_h^2}{\omega^4}}\right) \tag{36}$$

We see that k_r is complex (i.e., not totally propagating) when

$$\boxed{\omega < \sqrt{2k_h v_{A,r} N}} \tag{37}$$

where $k_h \simeq \sqrt{\ell(\ell+1)}/r$. This effect has been suspected to be the origin of suppressed dipole oscillation modes in a large minority of red giants (e.g., [13]).

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