MHD Dynamo

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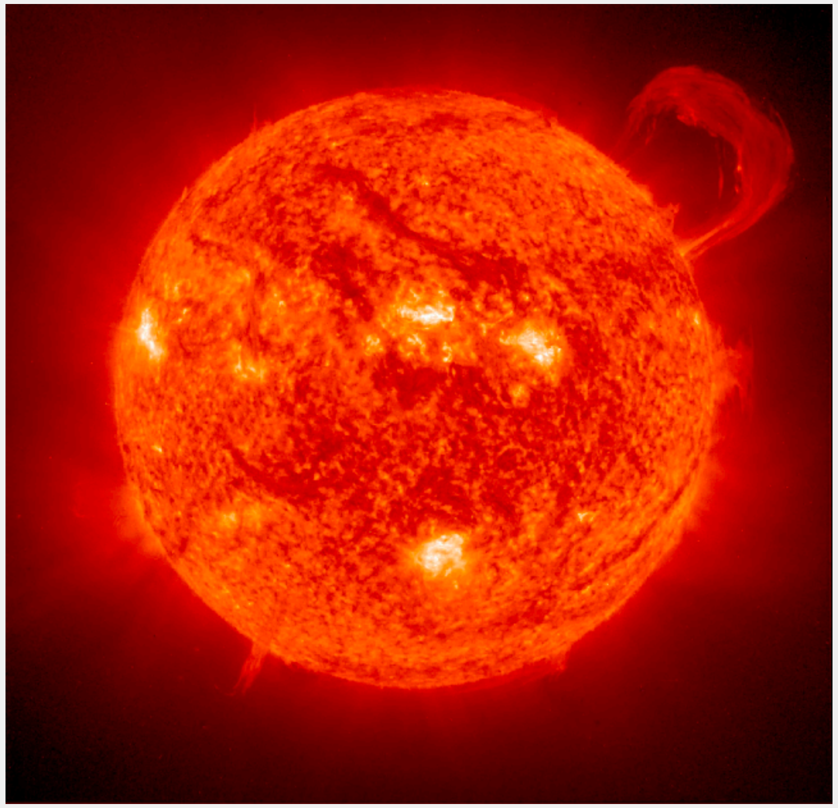
Astrophysical plasma

- Stars!
- ISM is at least partially ionized (charge neutrality overall)
- Magnetized
- No strong global electric fields
  - small-scale transient electric fields may exist



Ionized matter

Vlasov method

- Tracks distribution function $f(x, p)$
- most accurate

$$\frac{\partial f_e}{\partial t} + \mathbf{v}_e \cdot \nabla f_e - e\left(\mathbf{E} + \frac{\mathbf{v_e}}{c} \times \mathbf{B}\right) \cdot \frac{\partial f_e}{\partial \mathbf{p}} = 0$$
$$\frac{\partial f_i}{\partial t} + \mathbf{v}_i \cdot \nabla f_i + Z_i e\left(\mathbf{E} + \frac{\mathbf{v_i}}{c} \times \mathbf{B}\right) \cdot \frac{\partial f_i}{\partial \mathbf{p}} = 0$$

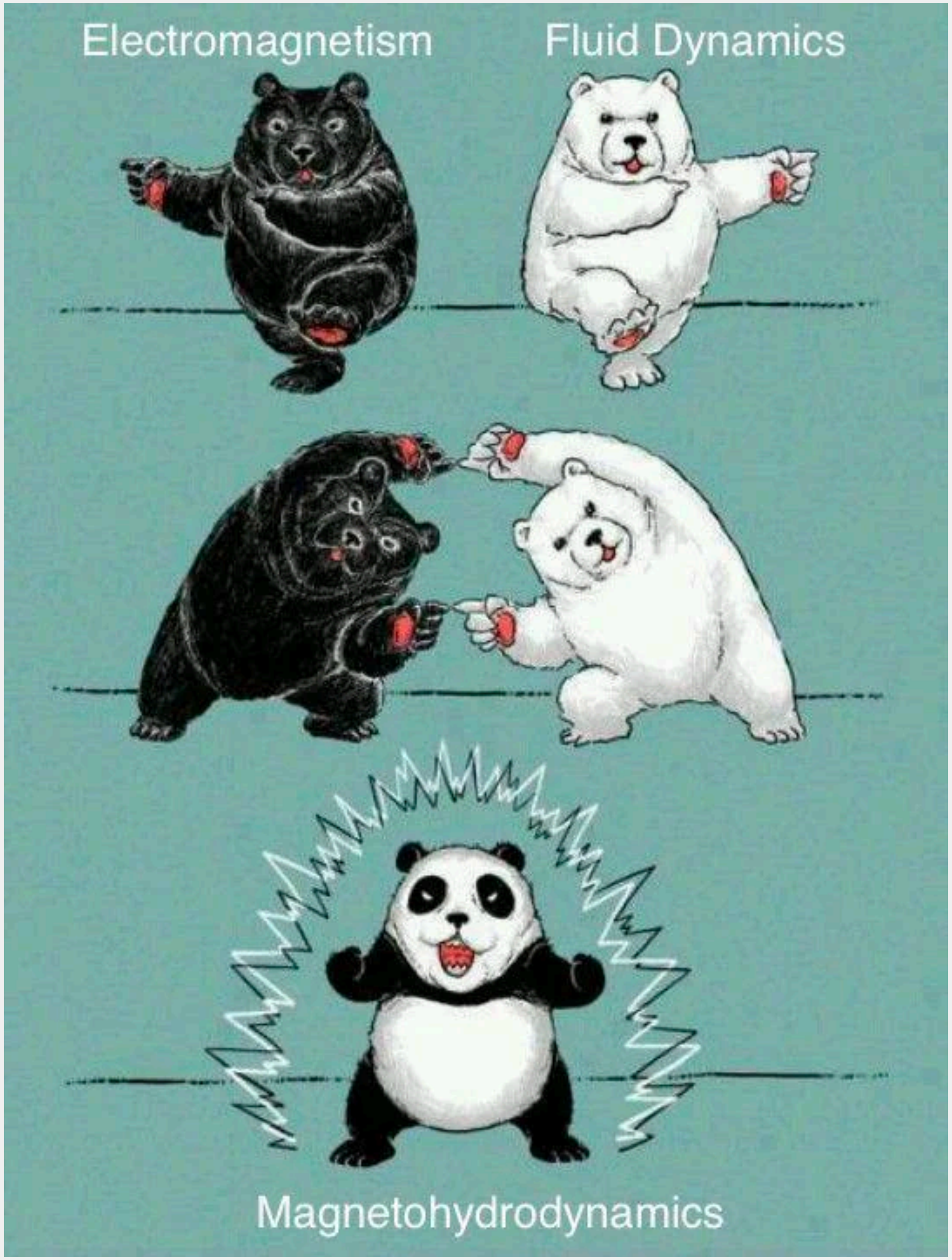
Two-fluid approach

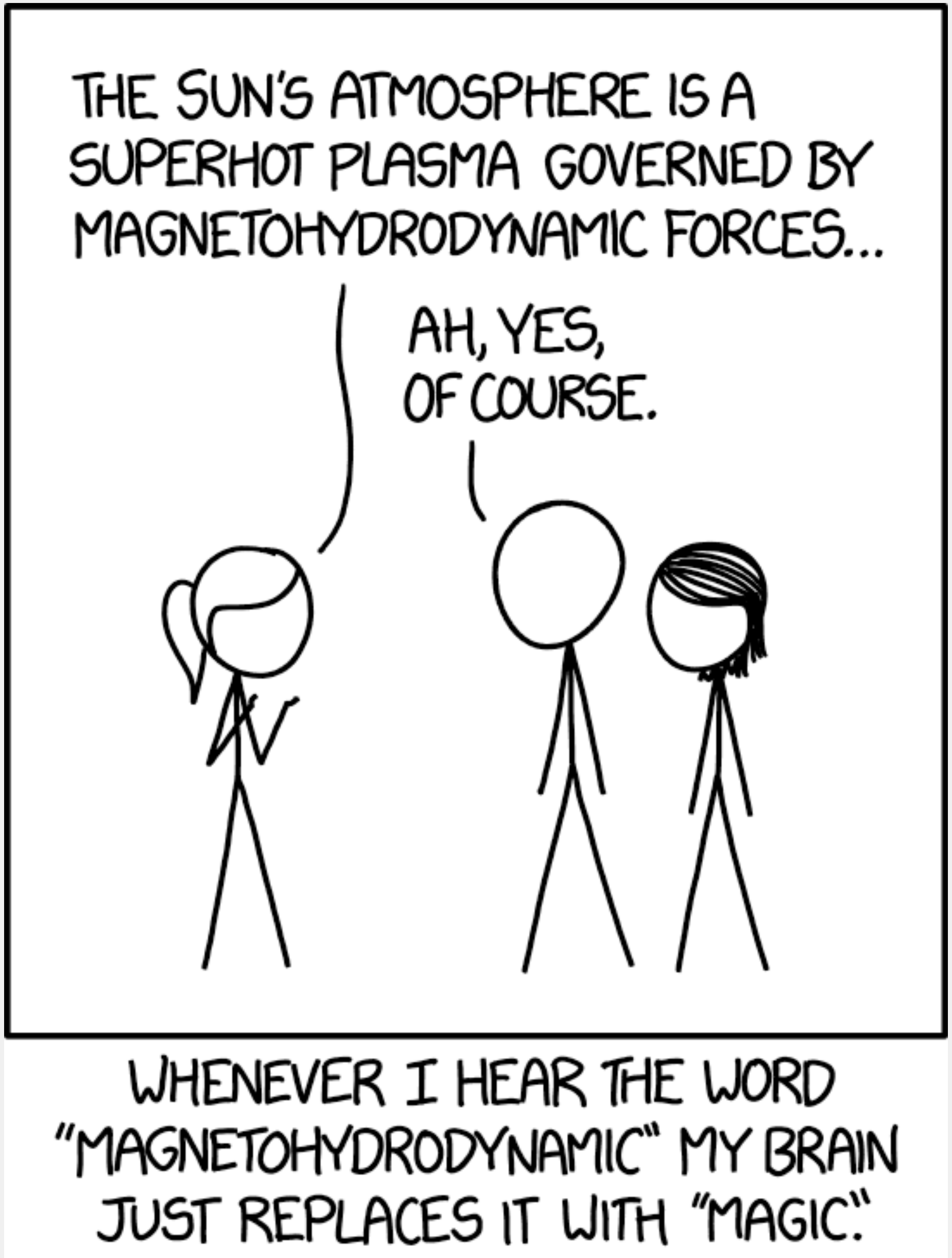
- full ionized plasma
- electrons + ions as two distinct fluids

$$m_i n_i \frac{\mathrm{D}\mathbf{u}_i}{\mathrm{D}t} = q_i n_i (\mathbf{E} + \mathbf{u}_i \times \mathbf{B}) + \nabla \cdot \hat{P}_i^{\top} - f_{ie} m_i n_i (\mathbf{u}_i - \mathbf{u}_e)$$
$$m_e n_e \frac{\mathrm{D}\mathbf{u}_e}{\mathrm{D}t} = q_e n_e (\mathbf{E} + \mathbf{u}_e \times \mathbf{B}) + \nabla \cdot \hat{P}_e^{\top} - f_{ei} m_e n_e (\mathbf{u}_e - \mathbf{u}_i)$$

Magnetohydrodynamics

- A single fluid model
- the least accurate


Electromagnetism
Fluid Dynamics
Magnetohydrodynamics


THE SUN'S ATMOSPHERE IS A SUPERHOT PLASMA GOVERNED BY MAGNETOHYDRODYNAMIC FORCES...
AH, YES, OF COURSE.
WHENEVER I HEAR THE WORD "MAGNETOHYDRODYNAMIC" MY BRAIN JUST REPLACES IT WITH "MAGIC."

MHD : a theory of *big* and *slow* plasma

1. Each species of particles are spatially well-mixed

   $L \gg \lambda_D, r_g^{(i,e)}$

2. Fluid motion is non-relativistic and slower than microscopic processes

   $T \gg \omega_p^{-1}, \omega_g^{(i,e)}$

- A good approximation in most astrophysical systems

ideal MHD

Maxwell equations

$$\nabla \cdot \mathbf{E} = 4\pi\rho_c$$

$$\nabla \cdot \mathbf{B} = 0$$

big and slow

$$\nabla \times \mathbf{B} = \frac{4\pi}{c}\mathbf{J} + \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t}$$

$$\nabla \times \mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t}$$

magnetic diffusivity

$$\longrightarrow \quad \frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{B}) + \frac{c^2}{4\pi\sigma}\nabla^2\mathbf{B}$$

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{B})$$

Ohm's law

$$\mathbf{E} = -\frac{\mathbf{v}}{c} \times \mathbf{B} + \frac{\mathbf{J}}{\sigma}$$

ideal MHD

Maxwell part

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{B})$$

fluid motion affects magnetic field evolution

$$\nabla \cdot \mathbf{B} = 0$$

Fluid part

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0$$

magnetic field affects fluid motion

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} = -\nabla P + \frac{1}{4\pi}(\nabla \times \mathbf{B}) \times \mathbf{B}$$

Flux freezing theorem

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{B})$$

- magnetic field lines are "frozen-in"

Problem 7.59 Prove **Alfven's theorem**: In a perfectly conducting fluid (say, a gas of free electrons), the magnetic flux through any closed loop moving with the fluid is constant in time. (The magnetic field lines are, as it were, "frozen" into the fluid.)

(a) Use Ohm's law, in the form of Eq. 7.2, together with Faraday's law, to prove that if $\sigma = \infty$ and $\mathbf{J}$ is finite, then

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{B}).$$

(b) Let $\mathcal{S}$ be the surface bounded by the loop ($\mathcal{P}$) at time t, and $\mathcal{S}'$ a surface bounded by the loop in its new position ($\mathcal{P}'$) at time $t + dt$ (see Fig. 7.56). The change in flux is

$$d\Phi = \int_{\mathcal{S}'} \mathbf{B}(t + dt) \cdot d\mathbf{a} - \int_{\mathcal{S}} \mathbf{B}(t) \cdot d\mathbf{a}.$$

Show that

$$\int_{\mathcal{S}'} \mathbf{B}(t + dt) \cdot d\mathbf{a} + \int_{\mathcal{R}} \mathbf{B}(t + dt) \cdot d\mathbf{a} = \int_{\mathcal{S}} \mathbf{B}(t + dt) \cdot d\mathbf{a}$$

(where $\mathcal{R}$ is the "ribbon" joining $\mathcal{P}$ and $\mathcal{P}'$), and hence that

$$d\Phi = dt \int_{\mathcal{S}} \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{a} - \int_{\mathcal{R}} \mathbf{B}(t + dt) \cdot d\mathbf{a}$$

(for infinitesimal dt). Use the method of Sect. 7.1.3 to rewrite the second integral as

$$dt \oint_{\mathcal{P}} (\mathbf{B} \times \mathbf{v}) \cdot d\mathbf{l},$$

and invoke Stokes' theorem to conclude that

$$\frac{d\Phi}{dt} = \int_{\mathcal{S}} \left(\frac{\partial \mathbf{B}}{\partial t} - \nabla \times (\mathbf{v} \times \mathbf{B}) \right) \cdot d\mathbf{a}.$$

Together with the result in (a), this proves the theorem.

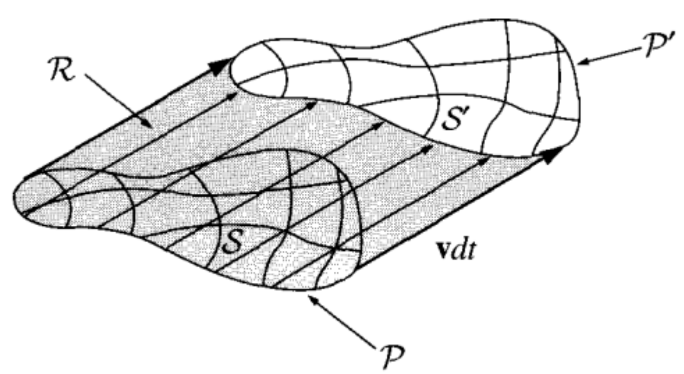
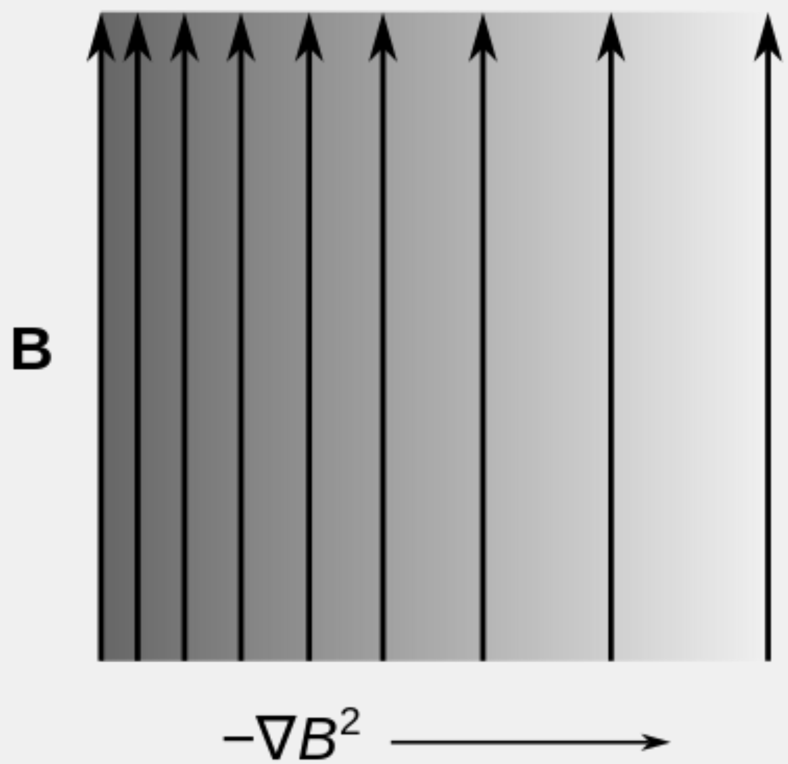


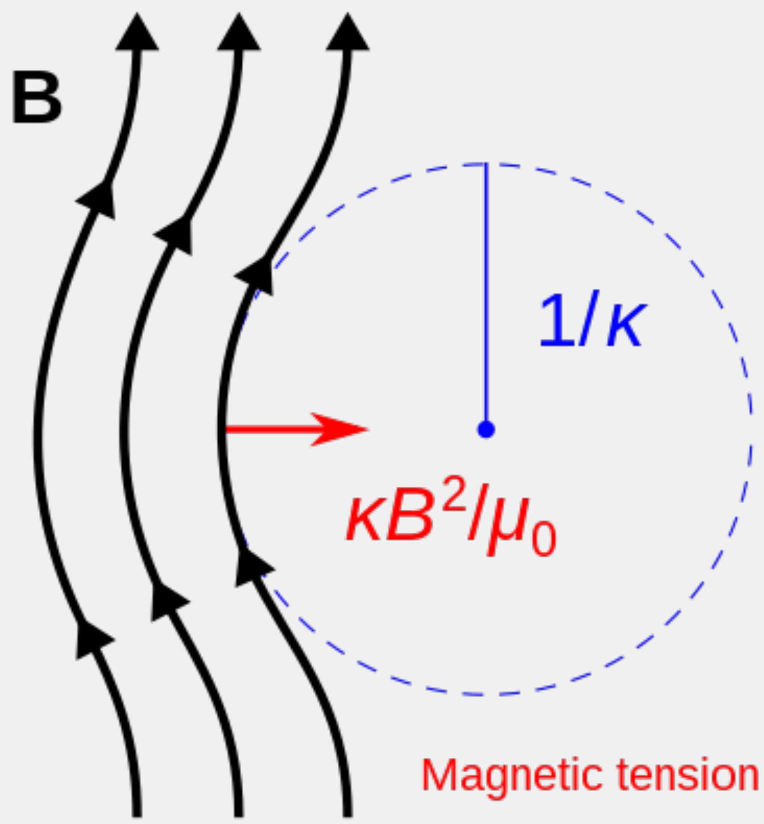
Figure 7.56

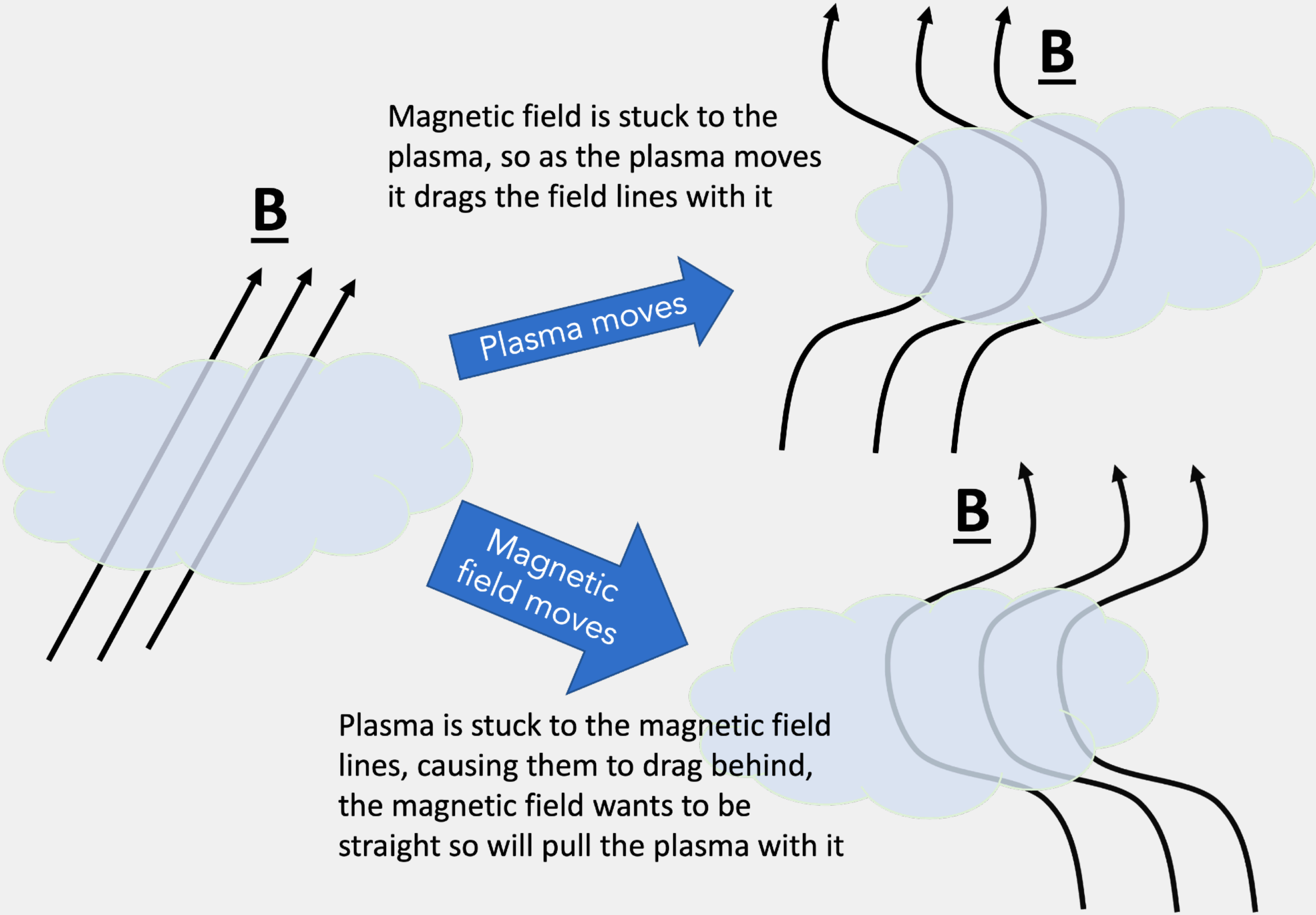
Magnetic effects on fluid motion

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} = -\nabla P + \frac{1}{4\pi}(\nabla \times \mathbf{B}) \times \mathbf{B}$$

- magnetic pressure
- magnetic tension







https://superdarn.ca/tutorials-14

Earth's magnetic field

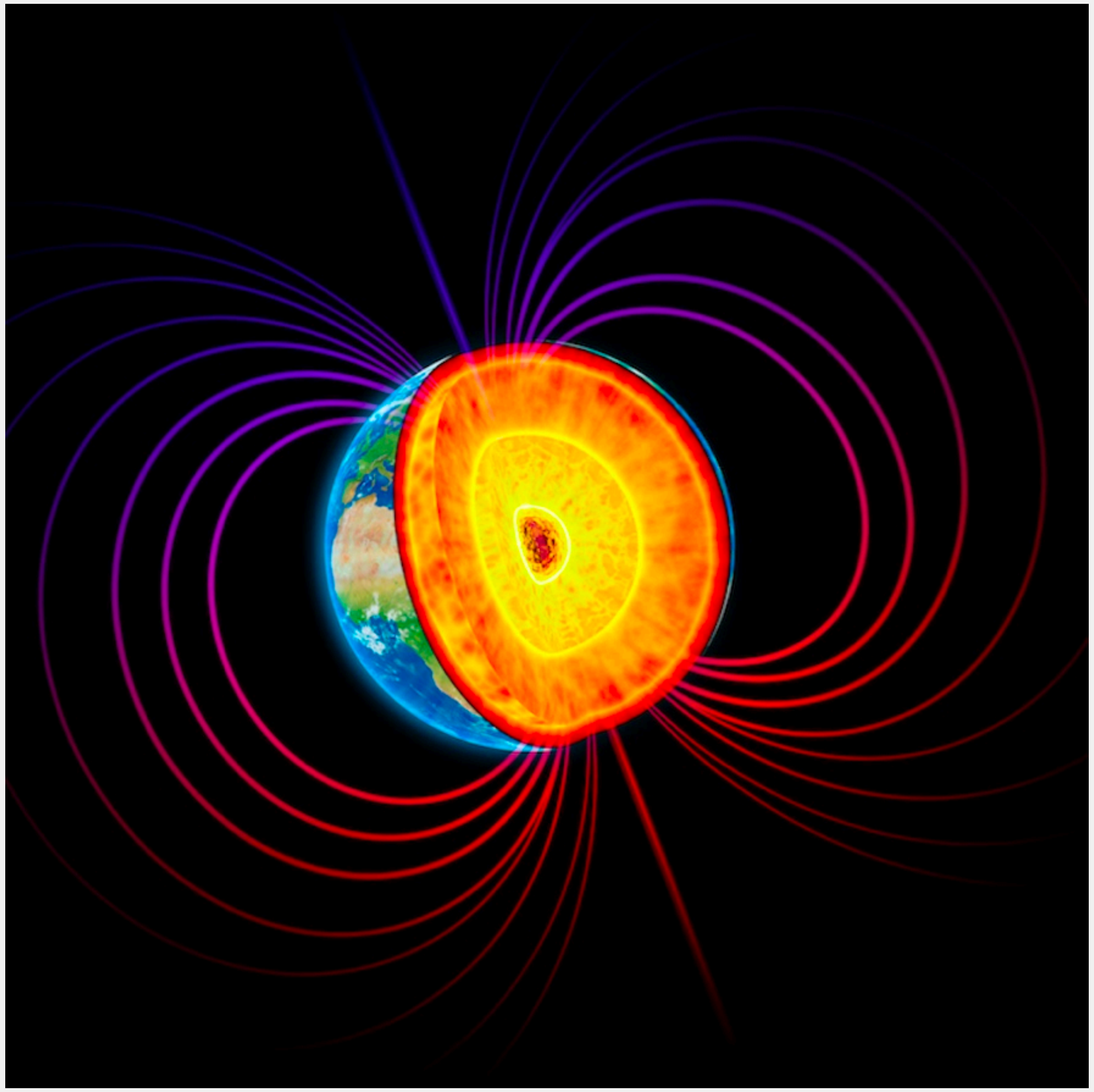
- Electrical resistance (finite conductivity) leads to diffusion

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{B}) + D_B \nabla^2 \mathbf{B}$$

$$D_B = \frac{c^2}{4\pi\sigma}$$

- Estimate from iron core : $\tau \sim \frac{L^2}{D_B}$

~3 Myr

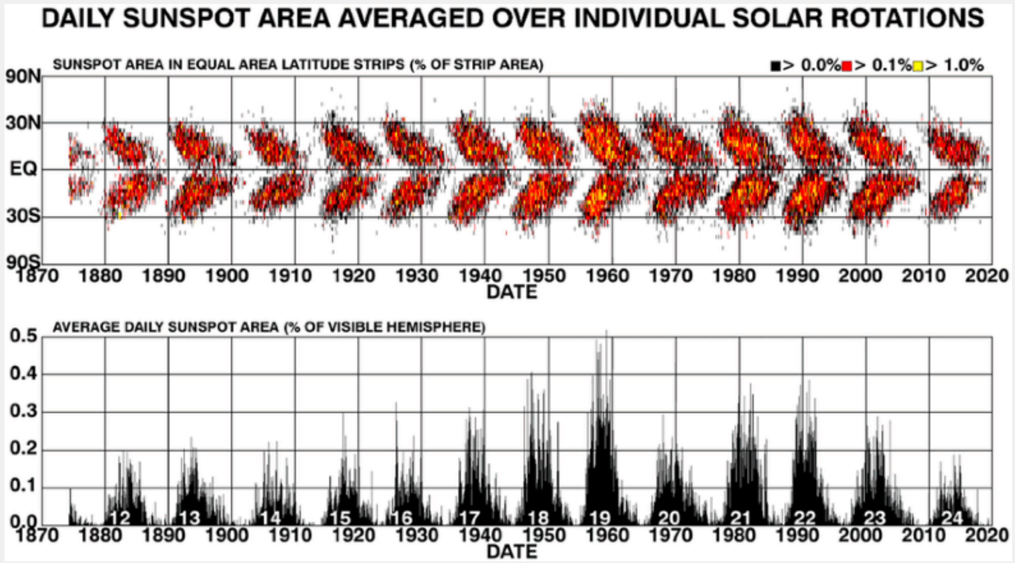


https://cosmosmagazine.com/earth/earth-sciences/core-conundrum-how-old-is-earths-magnetic-field/

some process should be regenerating the magnetic field

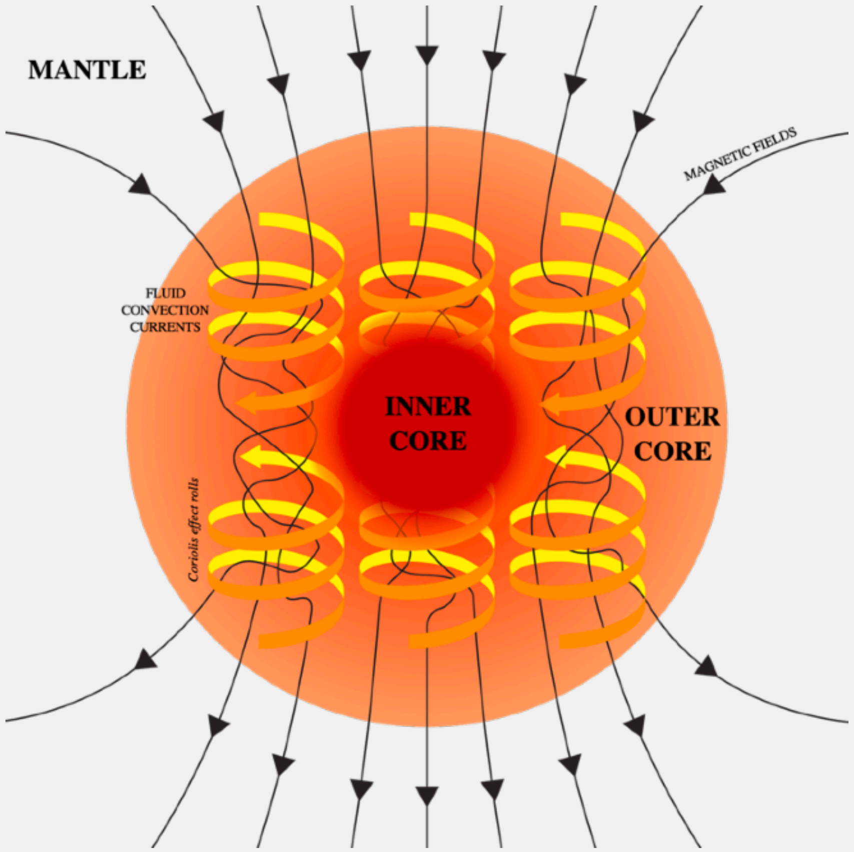
How does planets and stars have magnetic fields?

- Earth's age >> diffusion time scale
- Stellar magnetic fields
- 11-year cycle of sunspots..



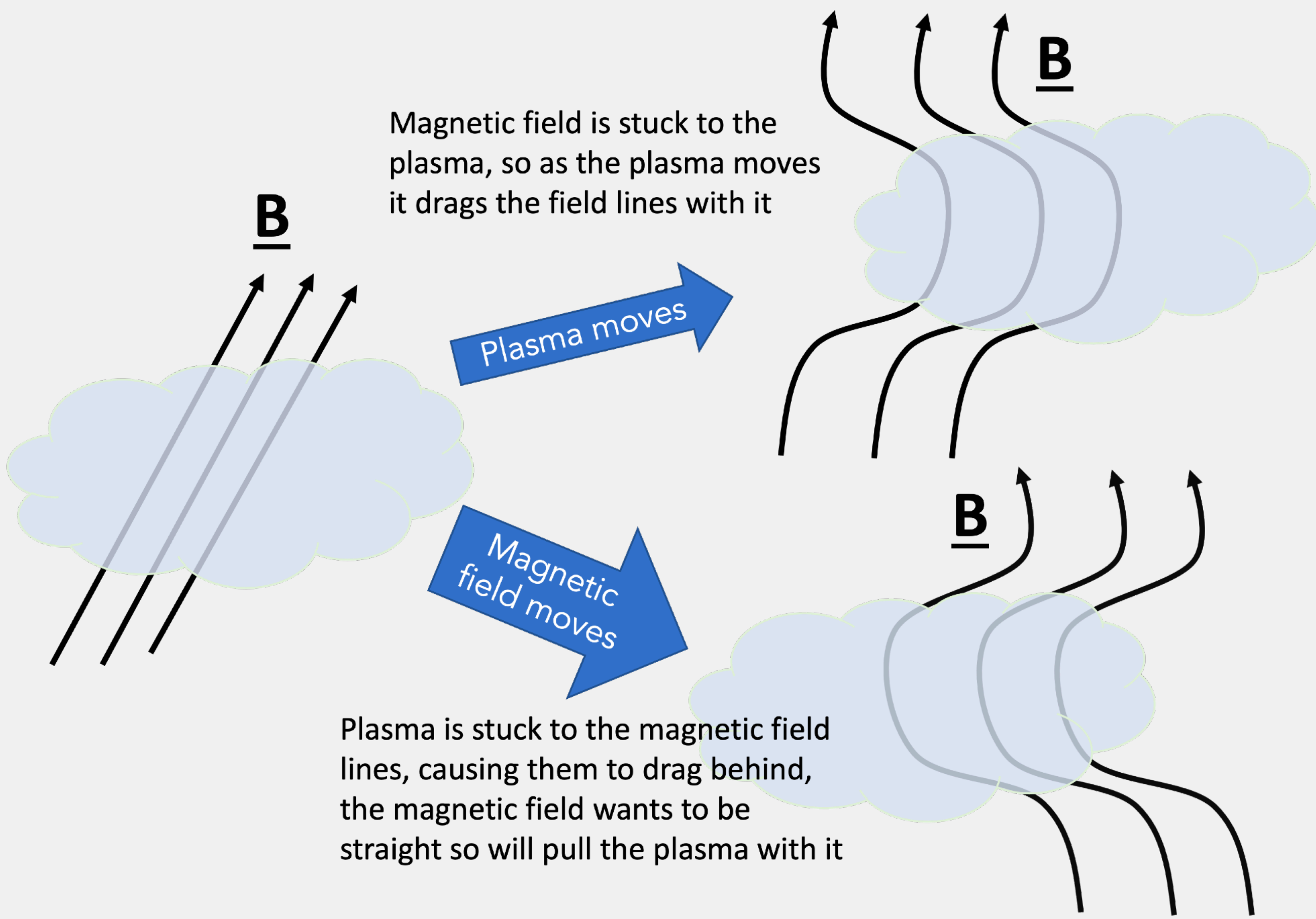
Dynamo theory

- Internal circulatory motions of conducting fluid compensates decay
- Convection, rotation, …



Wikipedia

recall) Fluid - Magnetic field interactions



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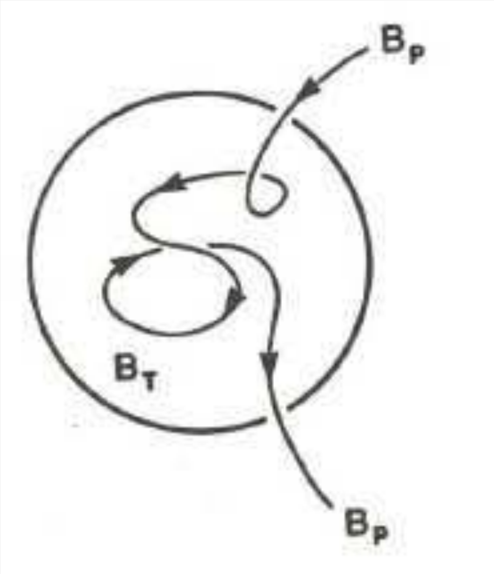
kinetic dynamo

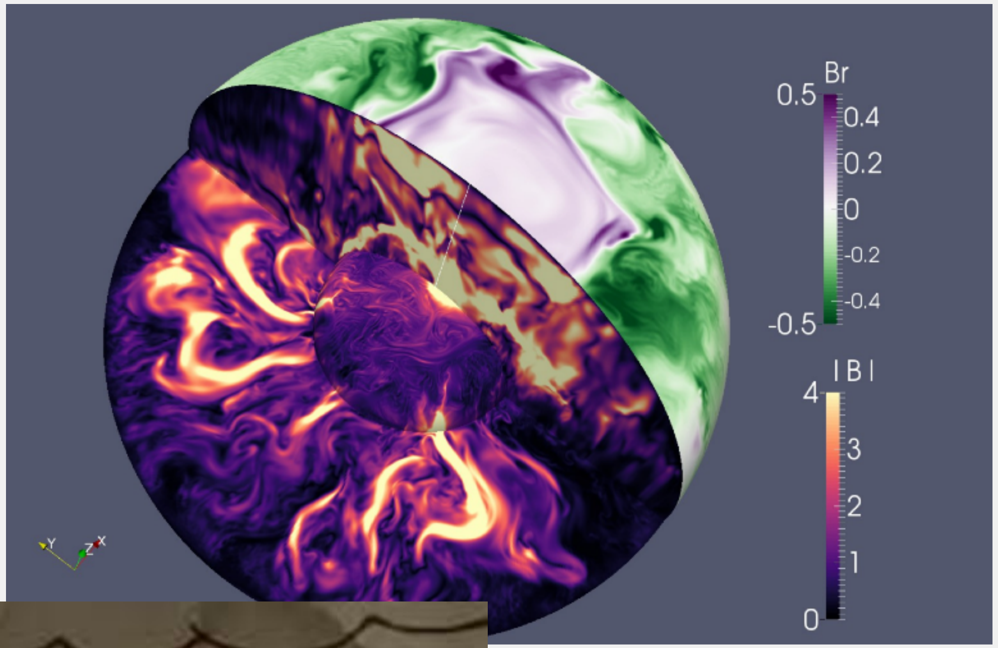
- For a given fluid flow $\mathbf{v}(x, y, z)$: how does $\mathbf{B}$ evolve and grow?
- Linear, eigenvalue problem



https://gfd.whoi.edu/wp-content/uploads/sites/18/2018/03/lecture1_136284.pdf

self-excited (dynamical) dynamo

- solve full MHD equations to get consistent $\mathbf{v}$ and $\mathbf{B}$
- nonlinear, turbulent
- usually requires numerical simulations



https://arxiv.org/pdf/1701.01299.pdf

solar / stellar dynamo

- agree with observations

- The first self-consistent dynamo models were developed in 1995

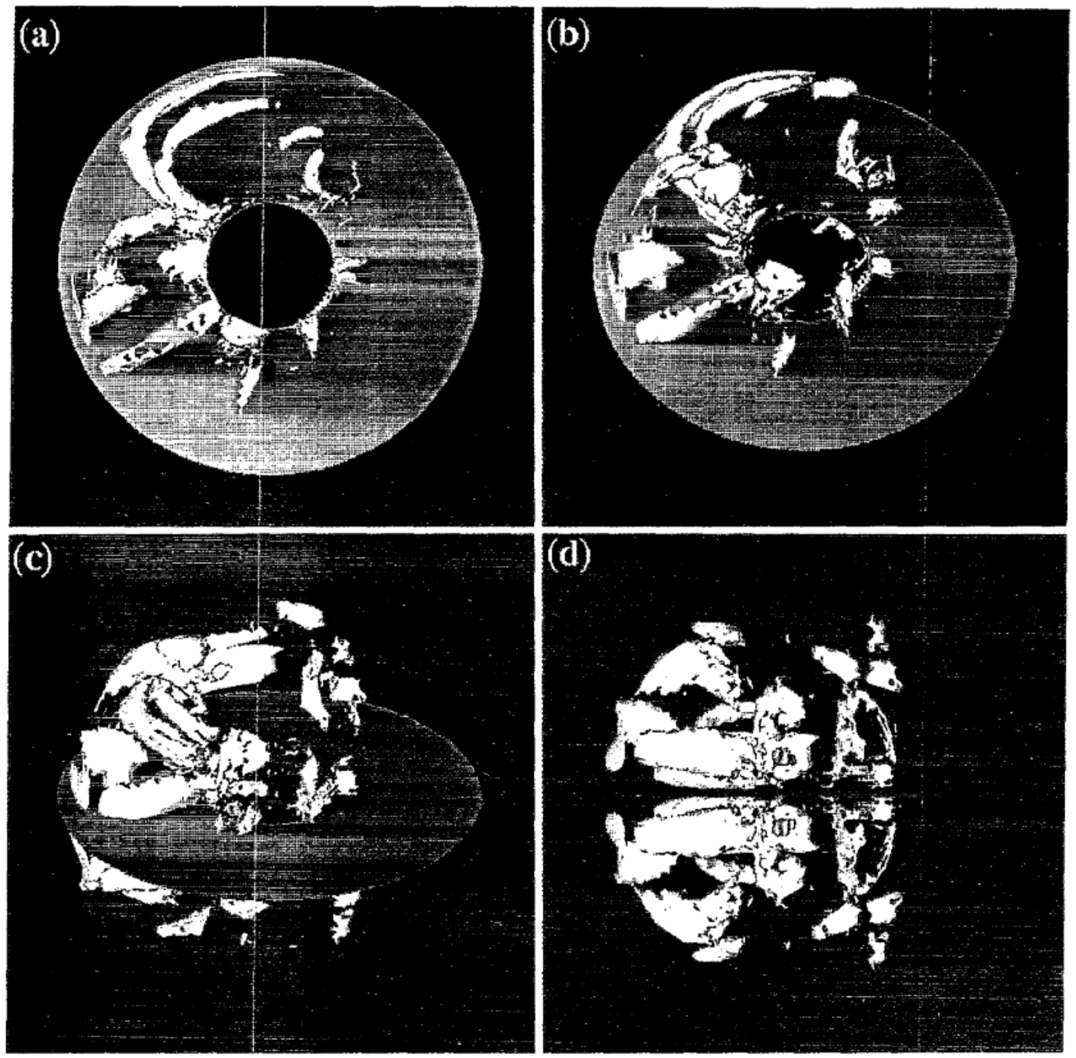
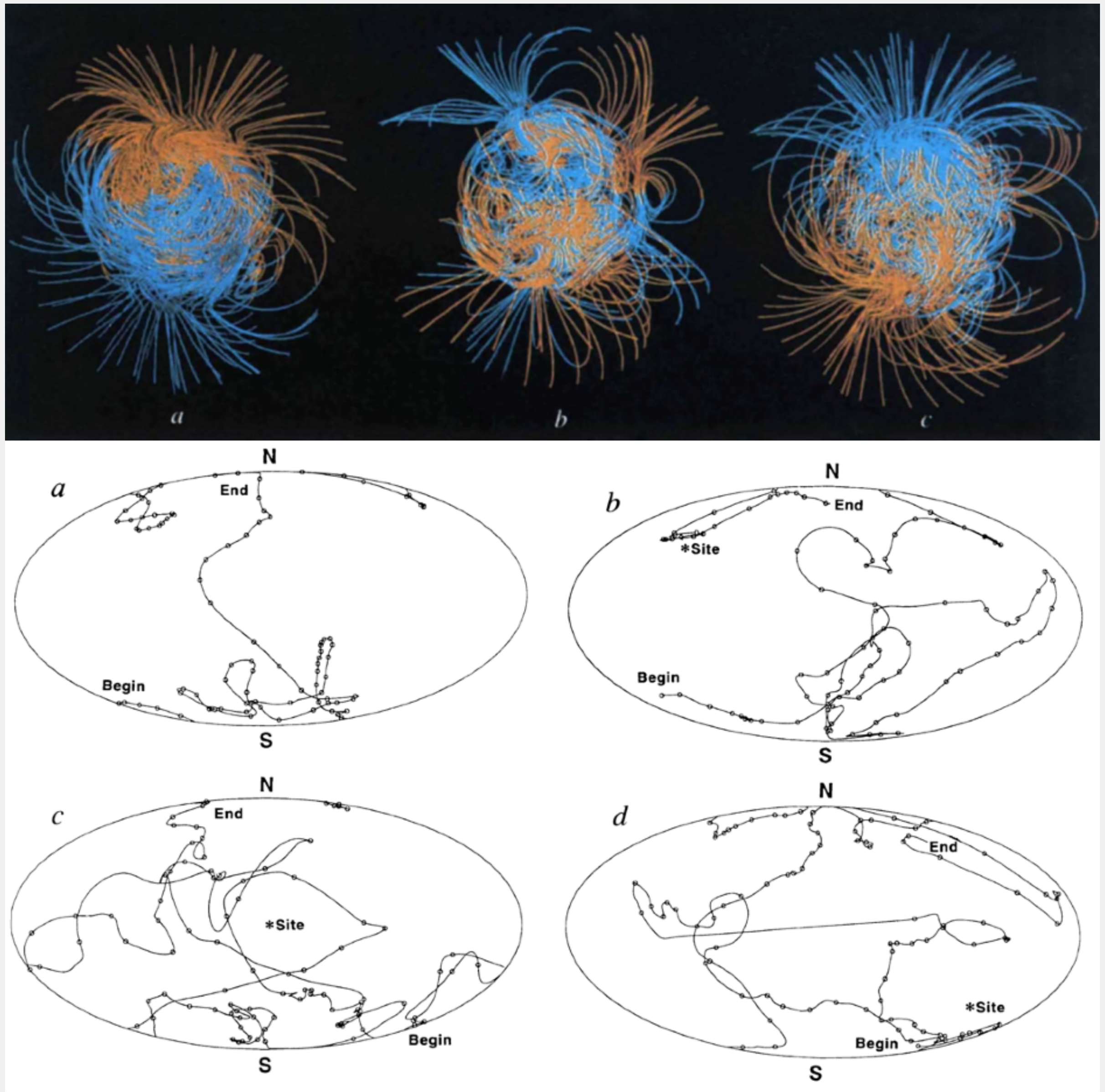


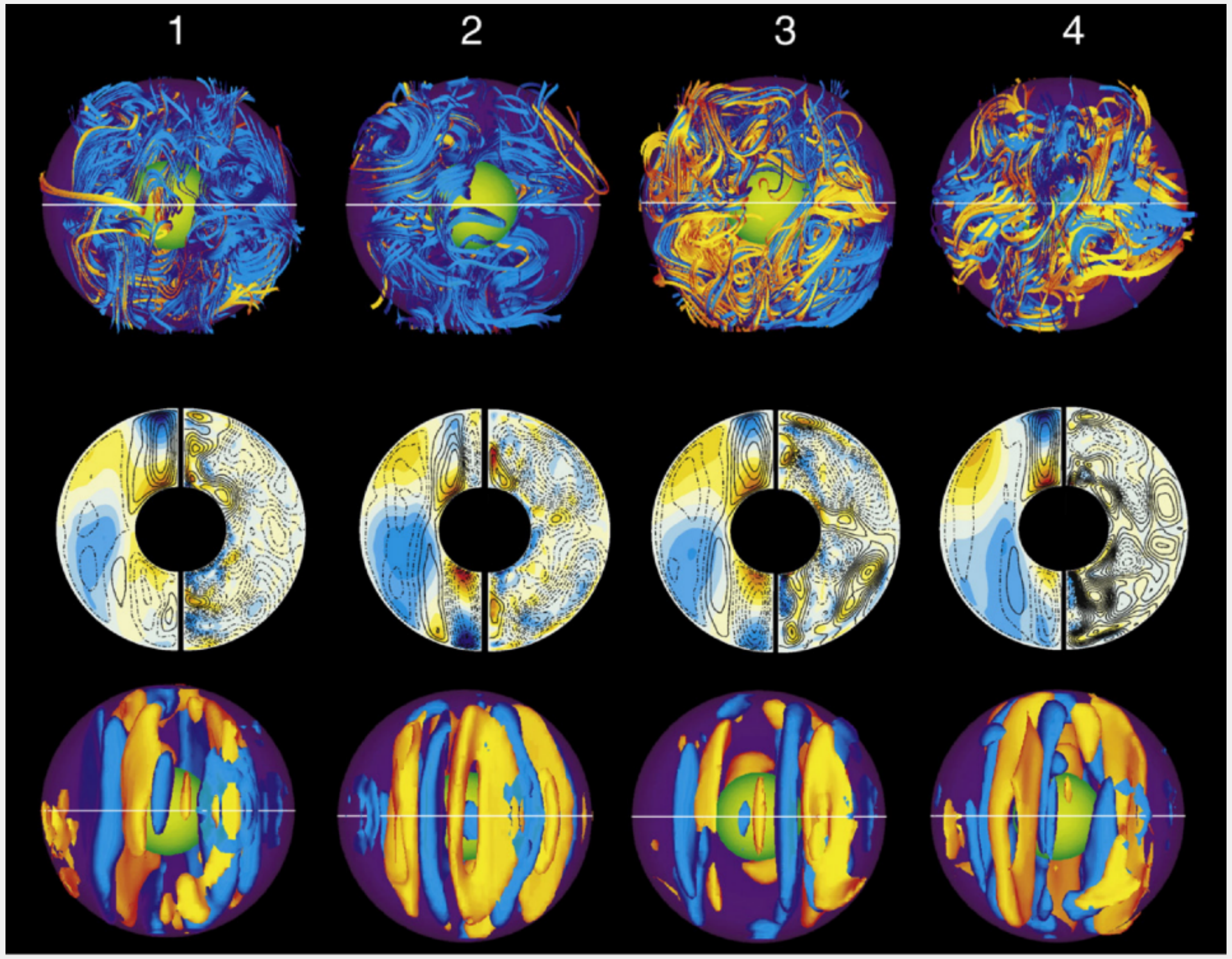
FIG. 9. The flux concentrated regions of the magnetic field perpendicular to the z axis visualized by the isosurface of $W_{\perp}=0.03$ when $\eta=1\times10^{-5}$. The meaning of the colors are the same as those in Fig. 8. Major parts of the isosurfaces is white. This means that these flux concentrated regions are out of convection columns.

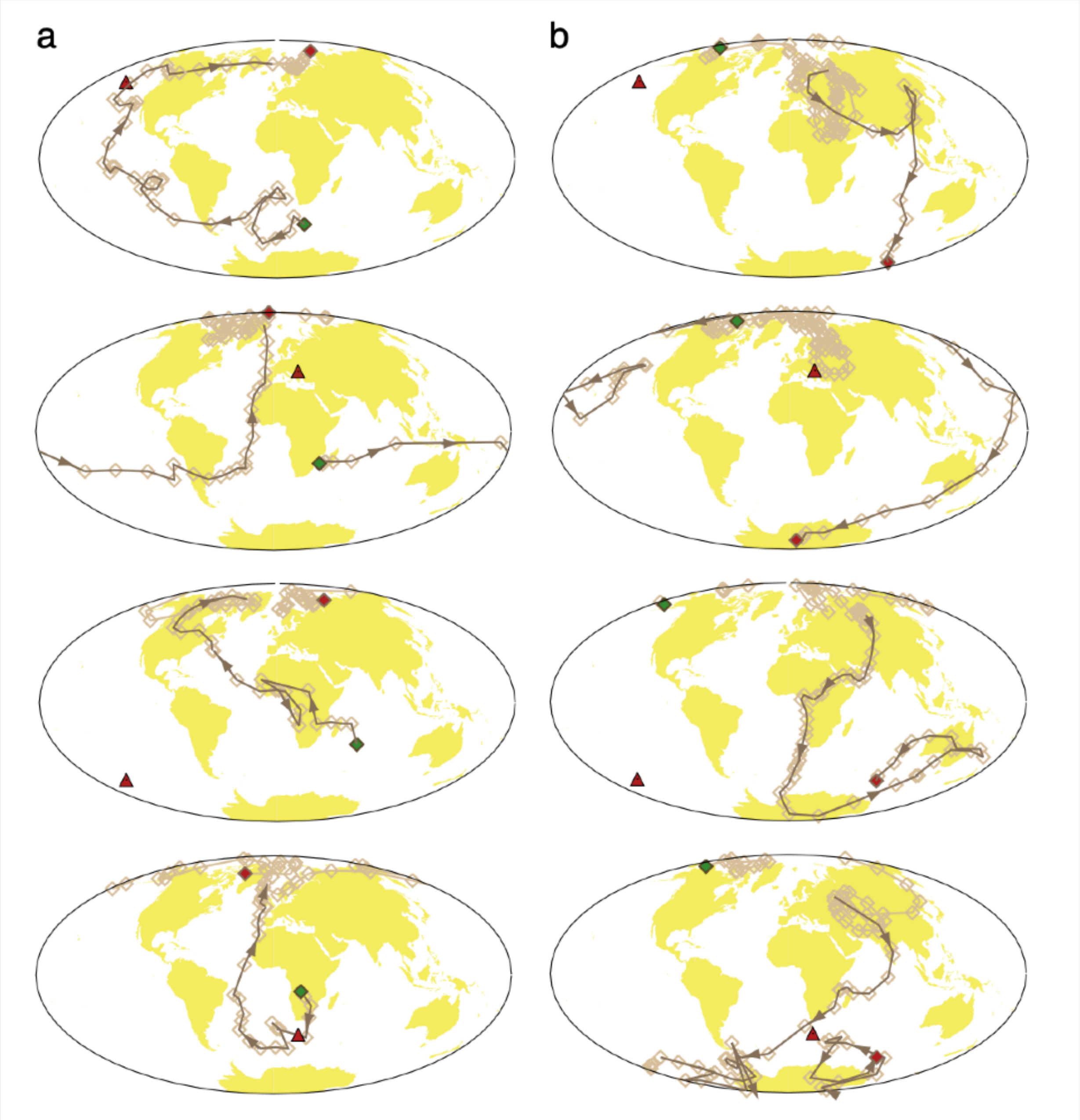
Kageyama & Sato 1995



Glatzmaier & Roberts 1995

- Modern geodynamo models successfully reproduce variation and polarity reversals





Olson+2011

Anti-dynamo theorems

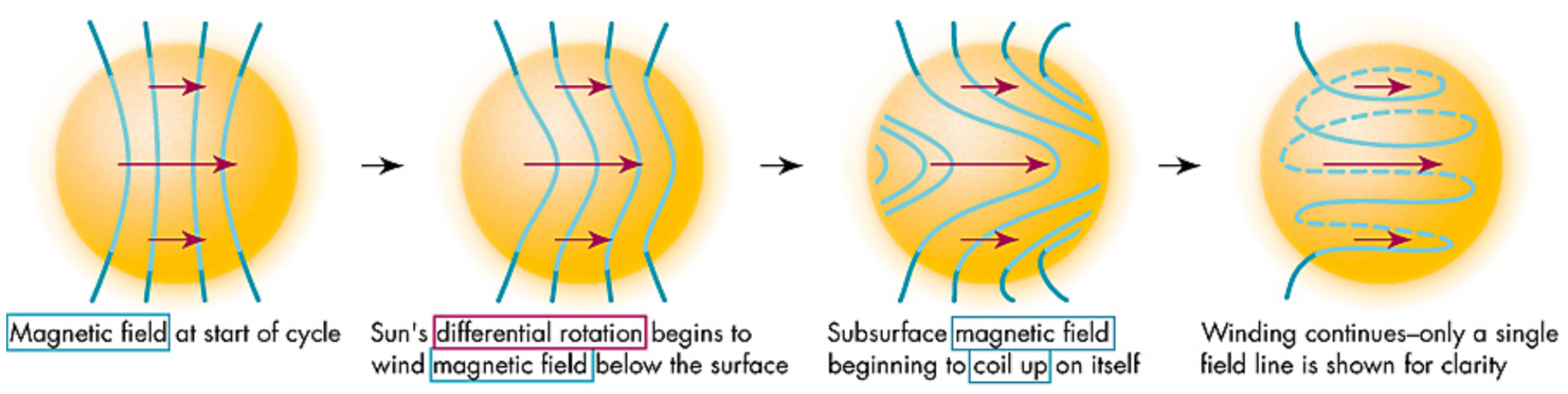
- Cowling's theorem : purely axisymmetric flow
- Zeldovich's theorem : 2D planar flow
- restricts type of magnetic field from dynamo origin

Successful dynamos cannot be too 'simple'

- usually require complex, 3D, nonlinear flows

Ω-effect

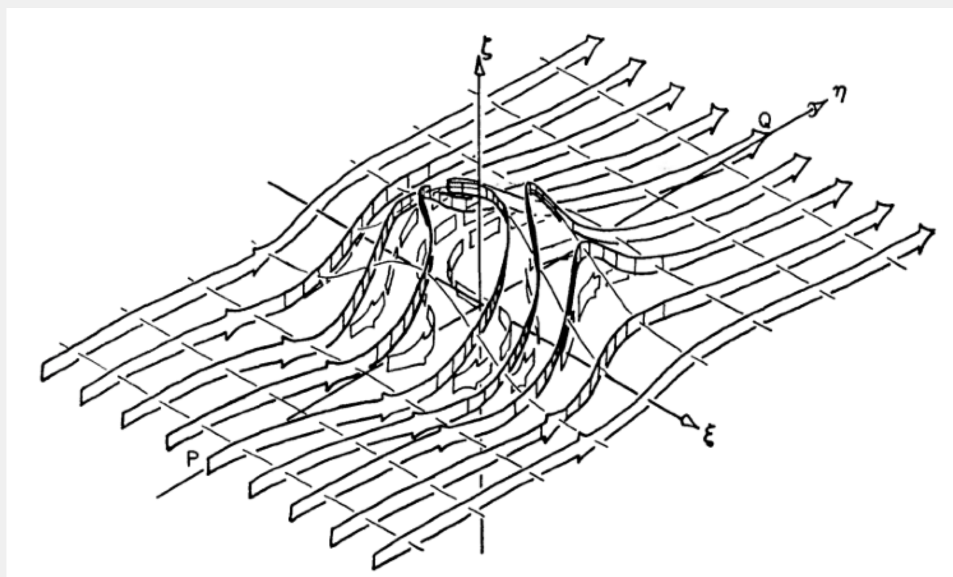
- Differential rotation converts meridional (poloidal) component to azimuthal (toroidal)
- winding up magnetic field lines



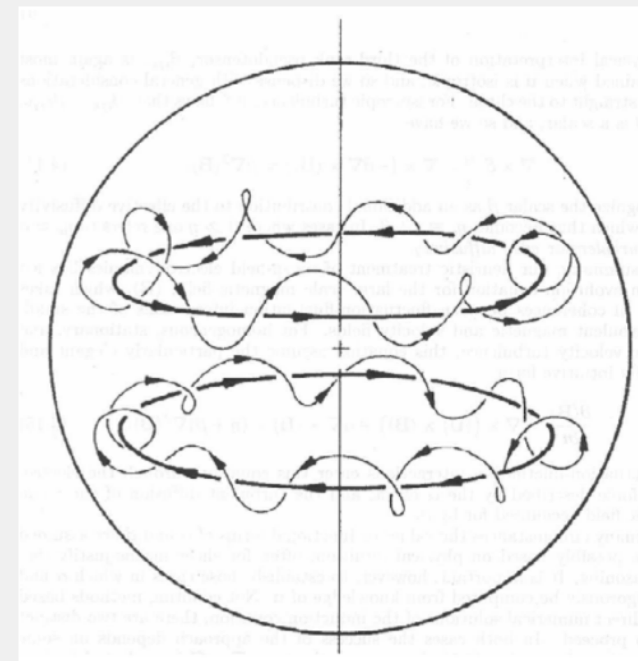
https://web.njit.edu/~cao/Phys320_L18.pdf

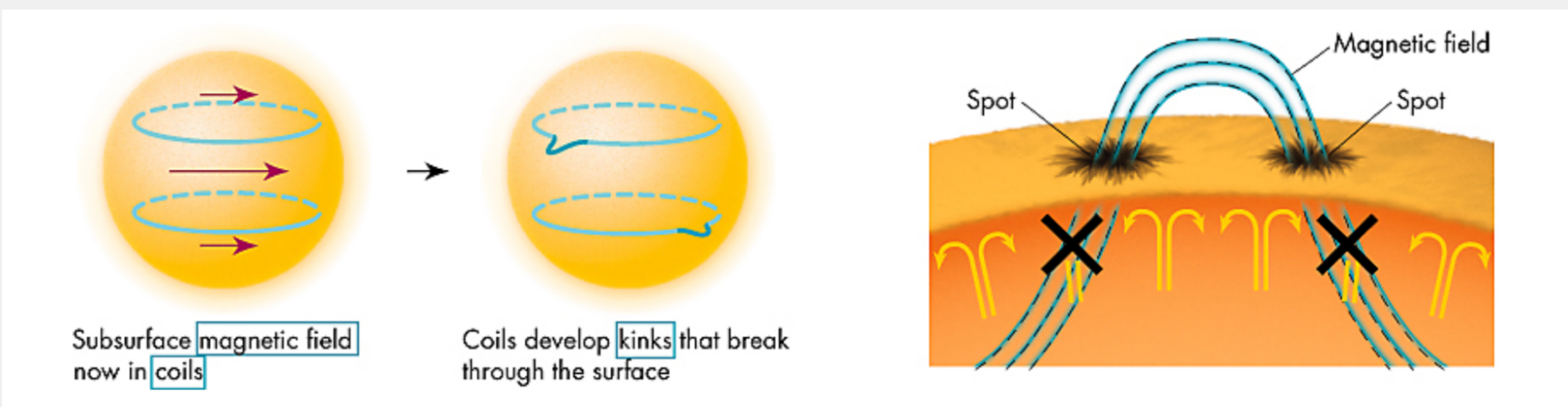
α-effect

- Coriolis force acting on ascending / descending fluid blobs (convection)
- Twisting of azimuthal field lines



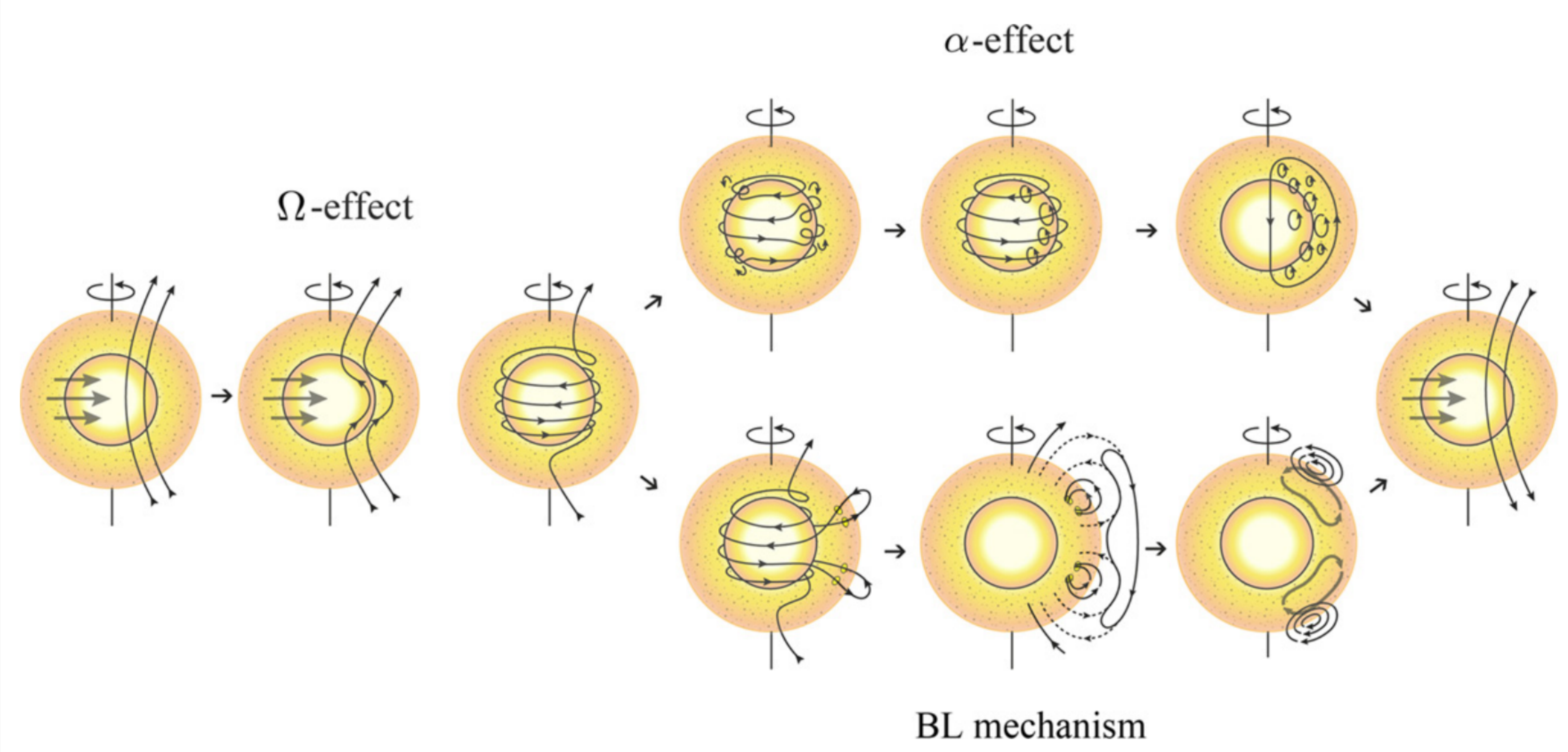
Parker 1955





https://web.njit.edu/~cao/Phys320_L18.pdf

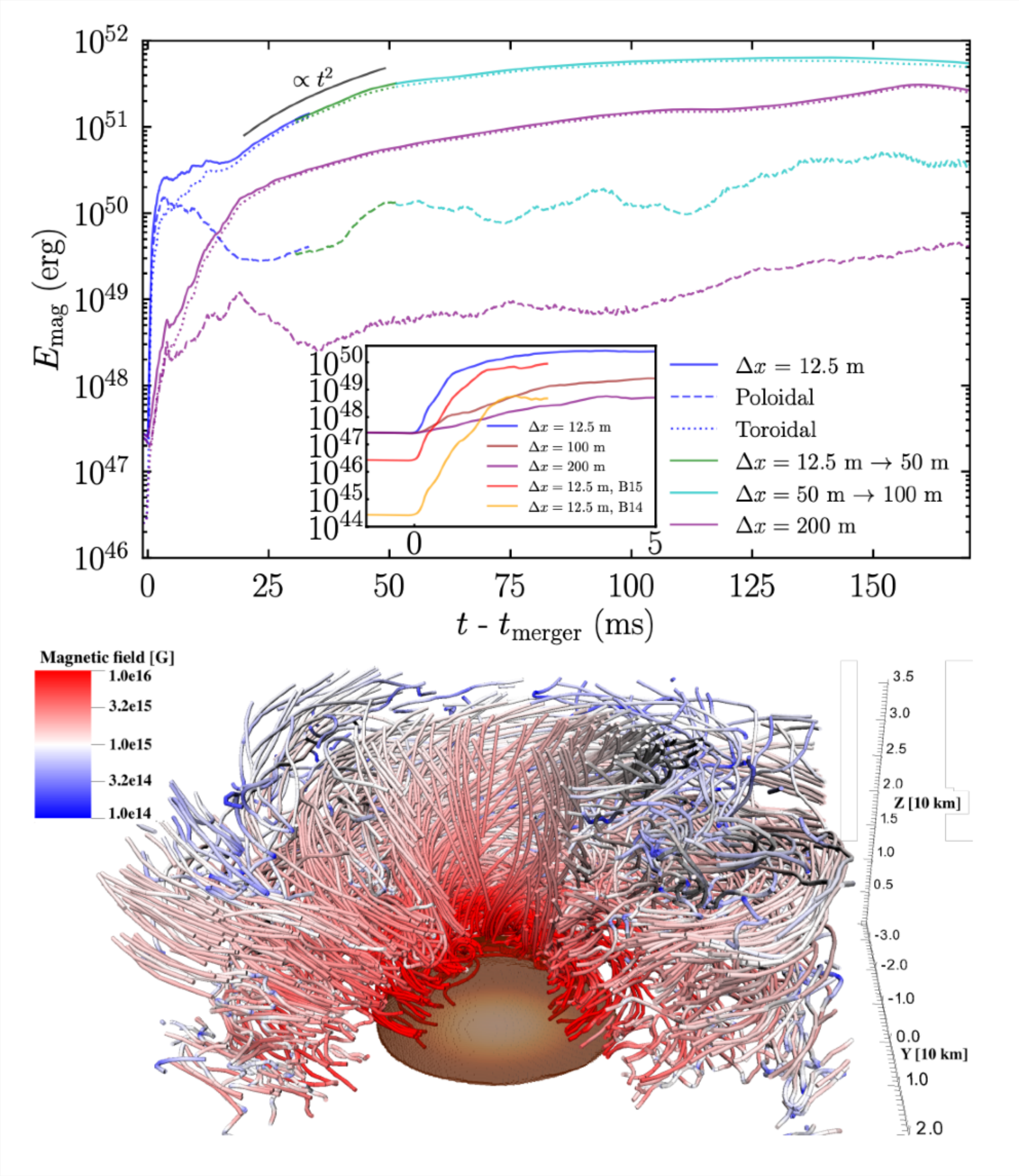
Example) Solar dynamo



Sanchez+2014

http://burro.cwru.edu/Academics/Astr221/StarPhys/SunSurface/dynamo.html

Dynamo action in BNS merger remnant



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