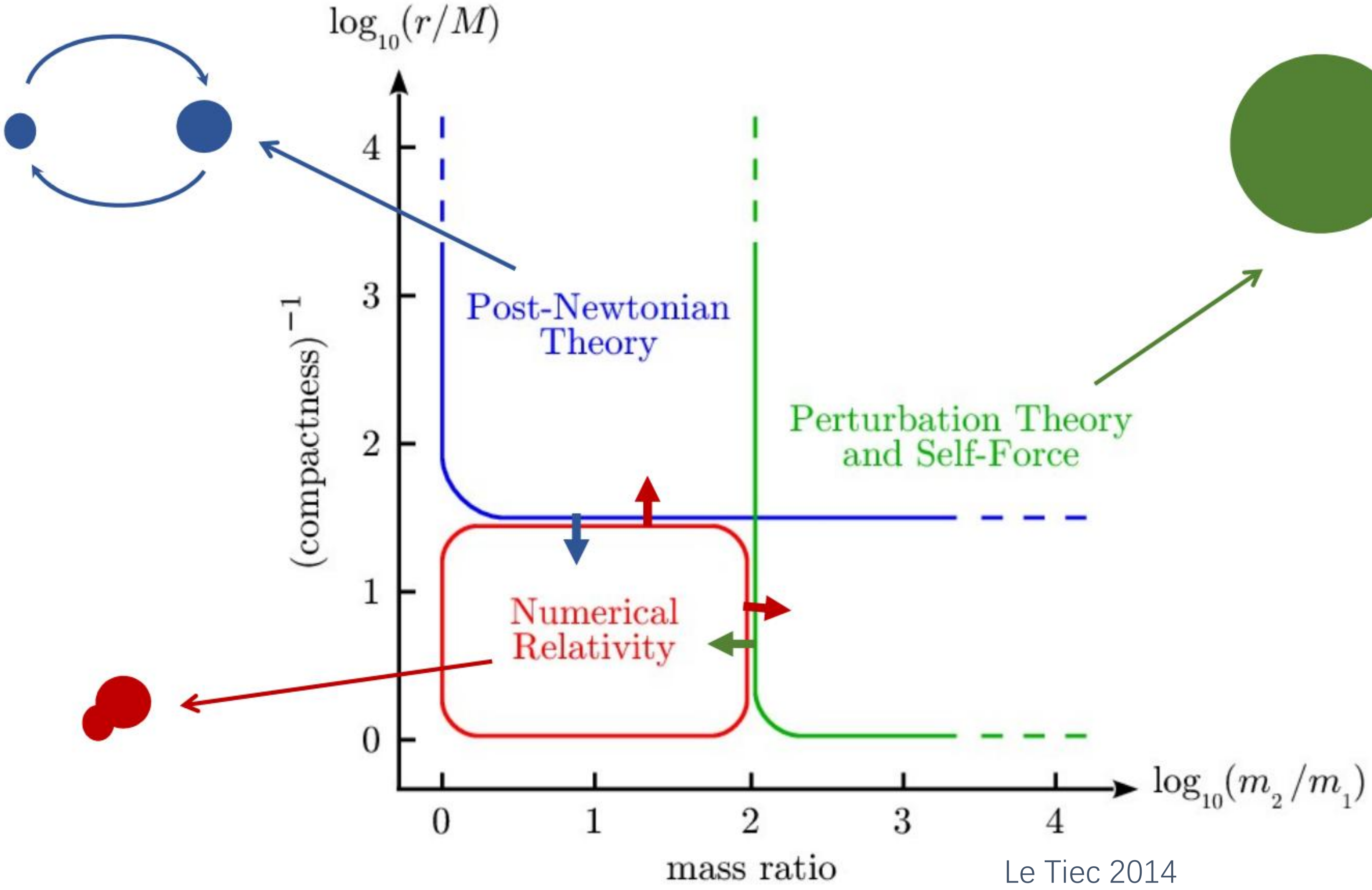
Caltech

Introduction to Post-Newtonian

Dongze Sun

ARC Seminar

June 10 2024



$\log_{10}(r/M)$
(compactness)$^{-1}$
0
1
2
3
4
Post-Newtonian Theory
Perturbation Theory and Self-Force
Numerical Relativity
$\log_{10}(m_2/m_1)$
mass ratio

Le Tiec 2014

Einstein's Equations

$$G_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}$$

- Introducing $h^{\alpha\beta} = \sqrt{-g}\,g^{\alpha\beta} - \eta^{\alpha\beta}$

$$\begin{cases} \Box h^{\mu\nu} = \dfrac{16\pi G}{c^4}\tau^{\mu\nu} \\ \partial_\nu h^{\mu\nu} = 0 \quad \text{(Harmonic gauge)} \end{cases}$$

$$\tau^{\alpha\beta} = \overset{\text{matter}}{|g| T^{\alpha\beta}} + \overset{\text{non-linearities}}{\frac{c^4}{16\pi G}\Lambda^{\alpha\beta}}$$

$$\begin{aligned} \Lambda^{\alpha\beta} = & -h^{\mu\nu}\partial^2_{\mu\nu}h^{\alpha\beta} + \partial_\mu h^{\alpha\nu}\partial_\nu h^{\beta\mu} + \frac{1}{2}g^{\alpha\beta}g_{\mu\nu}\partial_\lambda h^{\mu\tau}\partial_\tau h^{\nu\lambda} \\ & -g^{\alpha\mu}g_{\nu\tau}\partial_\lambda h^{\beta\tau}\partial_\mu h^{\nu\lambda} - g^{\beta\mu}g_{\nu\tau}\partial_\lambda h^{\alpha\tau}\partial_\mu h^{\nu\lambda} + g_{\mu\nu}g^{\lambda\tau}\partial_\lambda h^{\alpha\mu}\partial_\tau h^{\beta\nu} \\ & +\frac{1}{8}(2g^{\alpha\mu}g^{\beta\nu} - g^{\alpha\beta}g^{\mu\nu})(2g_{\lambda\tau}g_{\epsilon\pi} - g_{\tau\epsilon}g_{\lambda\pi})\partial_\mu h^{\lambda\pi}\partial_\nu h^{\tau\epsilon}. \end{aligned}$$

Einstein's Equations

$$\begin{cases} \Box h^{\mu\nu} = \dfrac{16\pi G}{c^4}\tau^{\mu\nu} \\ \partial_\nu \, h^{\mu\nu} = 0 \end{cases}$$

$$\Rightarrow \qquad h^{\alpha\beta} = \frac{16\pi G}{c^4}\Box^{-1}_{\mathrm{ret}}\tau^{\alpha\beta}$$

$$(\Box^{-1}_{\mathrm{ret}} f)(\mathbf{x}, t) \equiv -\frac{1}{4\pi}\iiint \frac{d^3\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|} f(\mathbf{x}', t - |\mathbf{x}-\mathbf{x}'|/c)$$

Post-Newtonian source

- Slowly moving: $v/c \ll 1,\ |T^{0i}/T^{00}| \ll 1$
- Weakly stressed: $|T^{ij}/T^{00}| \ll 1$
- Weakly gravitating: $|U/c^2| \ll 1$
- Isolated: $\lim_{\substack{r\to\infty \\ t+\frac{r}{c}=\mathrm{cst}}} \left(\frac{\mathrm{d}}{\mathrm{d}r} + \frac{\mathrm{d}}{c\mathrm{d}t}\right)\left(r\, h^{\alpha\beta}\right) = 0$
- Point particles (BBH)

Post-Newtonian source

$$\frac{dE(v)}{dt} = -F(v)$$

$$\Rightarrow \frac{dv}{dt}$$

Need $h_{\mu\nu}(v)$ to get E and F

Post-Newtonian Expansion

Slowly moving, Weakly stressed, Weakly gravitating

$$\Downarrow$$

$$h^{\mu\nu} = \sum_{n=2}^{+\infty} \frac{1}{c^n} h^{\mu\nu}_{(n)} \qquad \frac{\tau\left(\mathbf{x}', t - \frac{|\mathbf{x}-\mathbf{x}'|}{c}\right)}{|\mathbf{x}-\mathbf{x}'|} = \frac{\tau(\mathbf{x}', t)}{|\mathbf{x}-\mathbf{x}'|} - \frac{\dot{\tau}(\mathbf{x}', t)}{c} + \frac{|\mathbf{x}-\mathbf{x}'|}{2c^2}\ddot{\tau}(\mathbf{x}', t) + \cdots$$

$$\nabla^2 h^{\mu\nu}_{(n)} = 16\pi G \tau^{\mu\nu}_{(n-4)} + \partial_t^2 h^{\mu\nu}_{(n-2)}$$

- Isolated? Diverge as $|\mathbf{x}'| \to \infty$
- Point particles? Diverge as $\mathbf{x}' = \mathbf{x}_{1,2}$

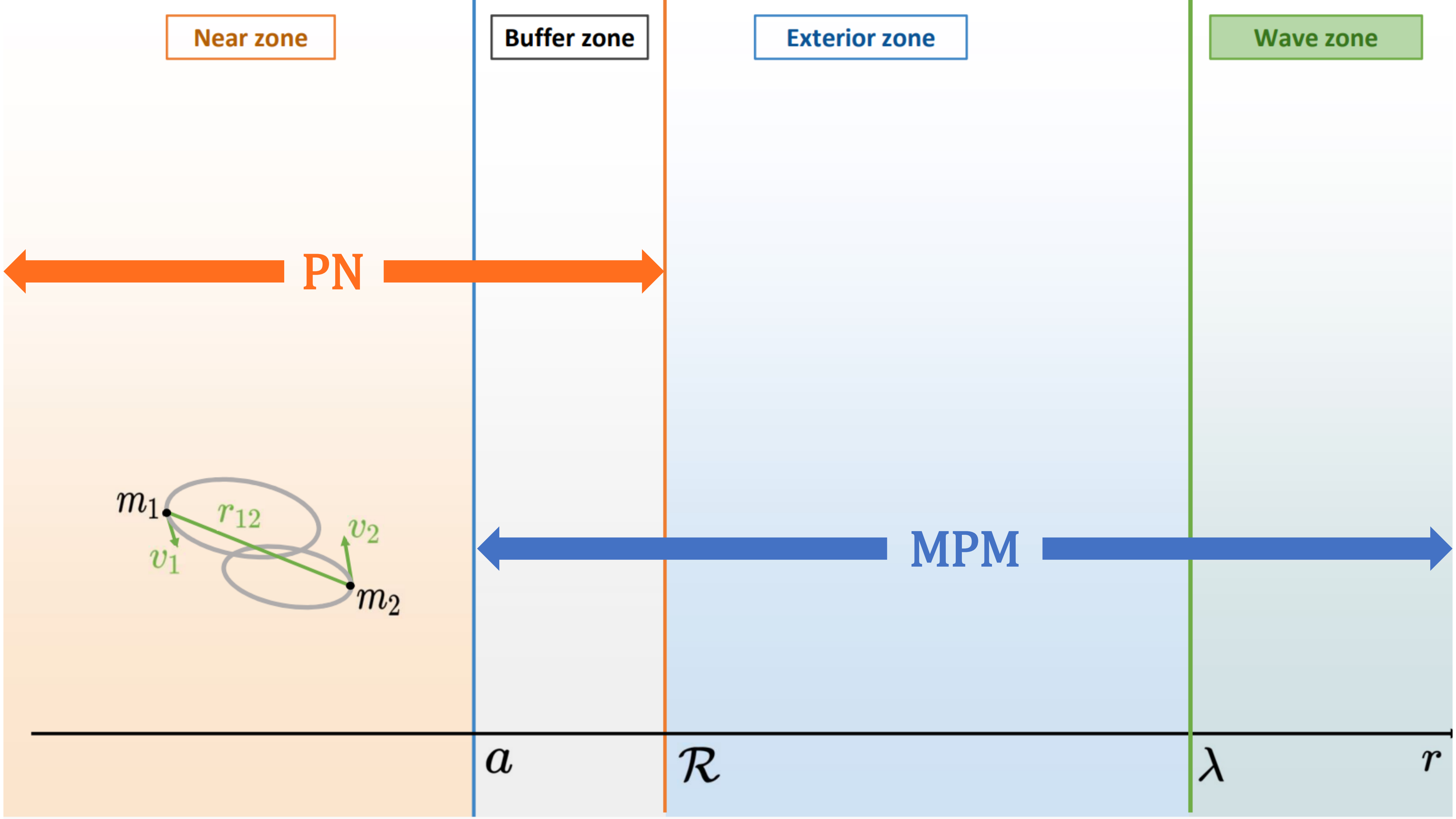
Multipolar Post-Minkowskian Expansion

$$\frac{\tau\left(\mathbf{x}', t - \frac{|\mathbf{x}-\mathbf{x}'|}{c}\right)}{|\mathbf{x}-\mathbf{x}'|} = \frac{\tau\left(\mathbf{x}', t - \frac{r}{c}\right)}{r} - x'^{j}\partial_j\left(\frac{\tau\left(\mathbf{x}', t - \frac{r}{c}\right)}{r}\right) + \cdots$$

- Weak gravity: $h_{\mathrm{ext}}^{\alpha\beta} = \sum_{n=1}^{+\infty} G^n h_n^{\alpha\beta}$

$$\Box h_n^{\alpha\beta} = \Lambda_n^{\alpha\beta}[h_1, h_2, \ldots, h_{n-1}]$$

- Isolated? ~~Diverge as $|\mathbf{x}'| \to \infty$~~ Diverge as $|\mathbf{x}'| \to 0$
- Point particles? Diverge as $\mathbf{x}' = \mathbf{x}_{1,2}$ (Hadamard regularization)



near zone
$r \leq \mathcal{R}$

buffer zone
$a < r \leq \mathcal{R}$

exterior zone
$r > a$

wave zone
$r \gg \lambda$

Bernard 2019

Matching

- Near zone:

$$h_{\rm in}^{\mu\nu} = \frac{16\pi G}{c^4}\Box^{-1}_{inst}\tau^{\mu\nu} - \frac{4G}{c^4}\sum_{l=0}^{\infty}\frac{(-)^l}{l!}\hat{\partial}_L\left[\frac{A_L^{\mu\nu}(t-\frac{r}{c}) - A_L^{\mu\nu}(t+\frac{r}{c})}{2r}\right]$$

- Exterior zone:

$$h_{(m)}^{\mu\nu}(t,x) = \Box^{-1}_{ret}\Lambda_{(m)}^{\mu\nu}(t,x) + \sum_{l=0}^{+\infty}\hat{\partial}_L\left(\frac{U_{(m)L}^{\mu\nu}(t-r/c)}{r}\right)$$

Multipole expansion$\left(h_{\rm in}^{\mu\nu}\right) \equiv$ PN expansion$\left(h_{\rm ext}^{\mu\nu}\right)$

$$\Rightarrow \quad U_{(m)}^{\mu\nu} = U_{(m)}^{\mu\nu}(\tau^{\mu\nu}, \mathrm{x}), \qquad A_L^{\mu\nu} = A_L^{\mu\nu}(U_{(m)}^{\mu\nu}, t)$$

radiation-reaction

Effective Field Theory

- Doesn't care about the detail about the short distance physics (near zone)

$$\exp\left(iS_{eff}[\phi]\right) = \int \mathcal{D}\Phi(x) e^{iS[\phi,\Phi]}$$

Radiation field $k_\mu \sim (v/r, v/r)$

Potential field $k_\mu \sim (v/r, 1/r)$

Or:

Particle x_μ

Gravitional field $h_{\mu\nu}$

$$S = S_{EH} + S_{pp}, \quad S_{pp} = -\sum_a m_a \int d\tau_a + \sum_a c_R^{(a)} \int d\tau_a R(x_a) + \sum_a c_V^{(a)} \int d\tau_a R_{\mu\nu}(x_a) \dot{x}_a^\mu \dot{x}_a^\nu + \cdots$$

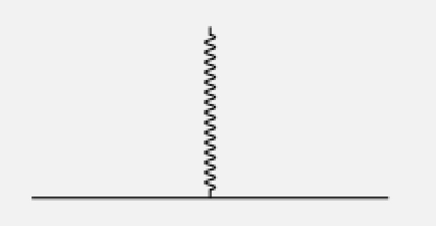
- No need for matching!

Effective Field Theory: 1PN

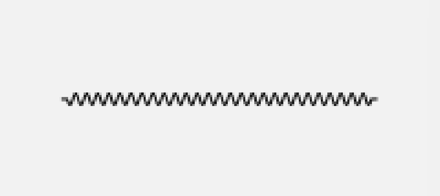
$$\exp\left[iS_{eff}(x_a)\right] = \int \mathcal{D}h(x)\exp\left[iS_{EH}(h) + iS_{pp}(h, x_a)\right]$$

$$\begin{aligned} S_{pp} &= -\sqrt{G_N}m\int dt\left(h_{00}/2 + v_i h_{0i} + v^i v^j h_{ij}/2 + \sqrt{G_N}h_{00}^2 \ldots\right) \\ S_{EH} &= \int d^4x\left[(\partial_i h)^2 - (\partial_t h)^2 + \sqrt{G_N}h(\partial h)^2 + \ldots\right] \end{aligned}$$

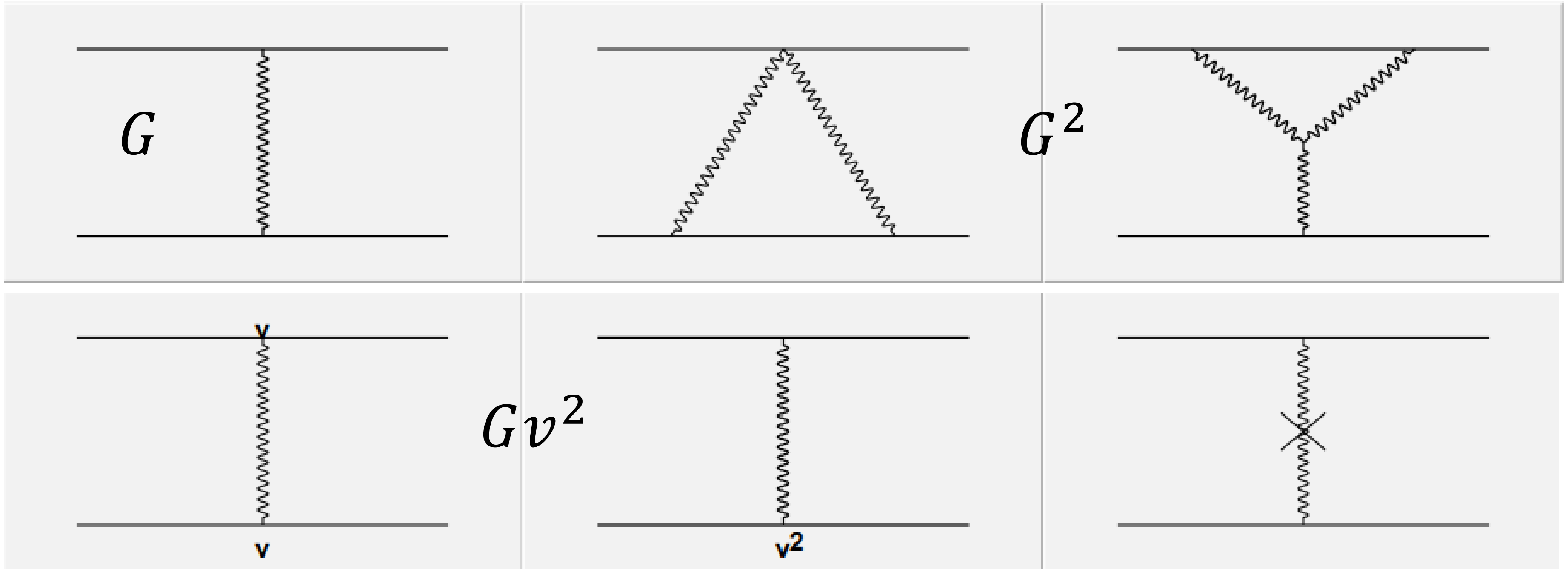
h-M Vertex: $\sim dt\, d^3k\sqrt{G_N}m$



Propagator: $\delta(t)\delta^{(3)}(k)\frac{1}{k^2}(1 + \frac{k_0^2}{k^2} + \ldots)$



Effective Field Theory: 1PN

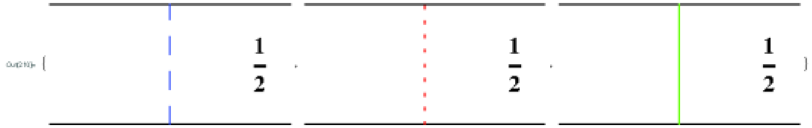


Sturani 2012

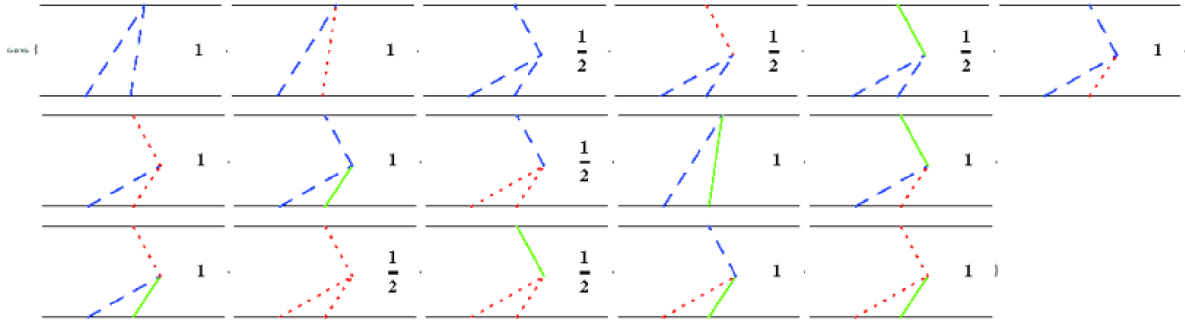
$$V = -\frac{Gm_1m_2}{2r}\left[1-\frac{G_Nm_1}{2r}+\frac{3}{2}(v_1^2)-\frac{7}{2}v_1v_2-\frac{1}{2}v_1\hat{r}v_2\hat{r}\right]+ \\ 1\leftrightarrow 2$$

Effective Field Theory: 3PN

Gv^6

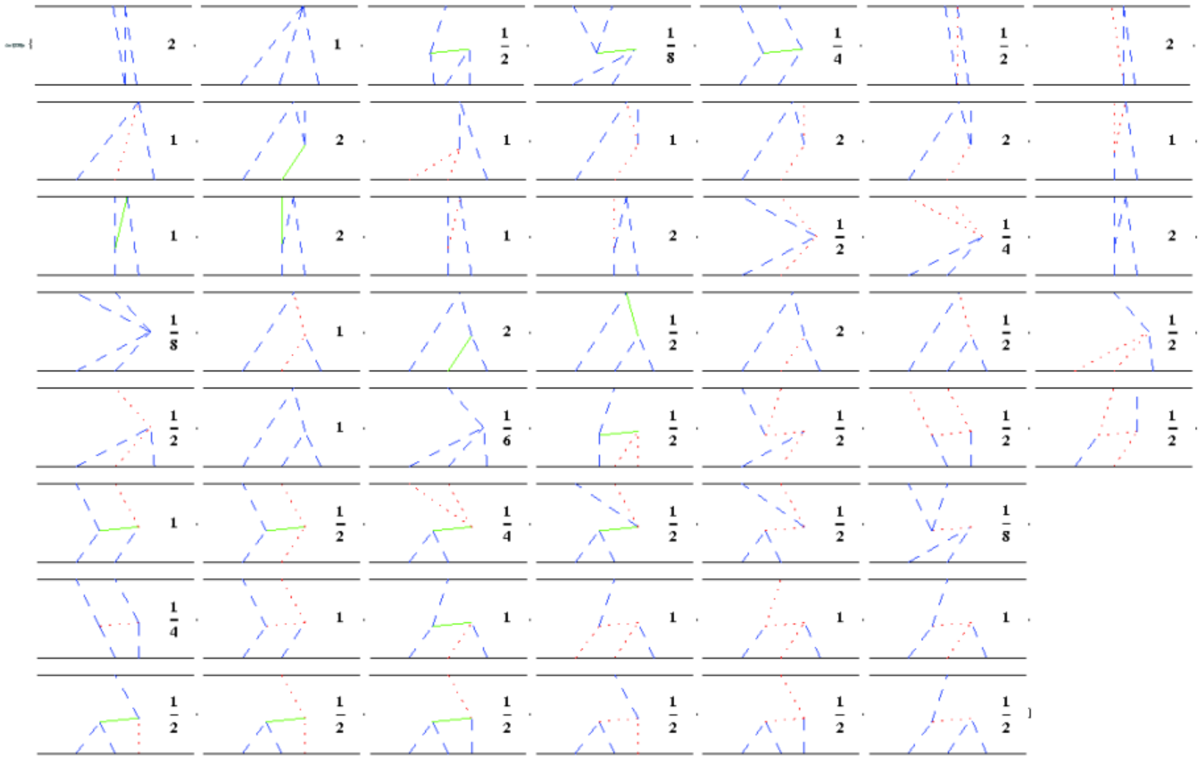


G^2v^4

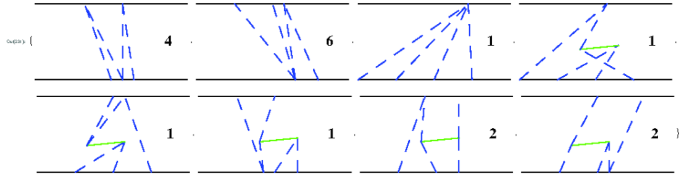


Sturani 2012

G^3v^2



G^4



Effective Field Theory: 4PN

- 3 @ Gv^8
- 23 @ G^2v^6
- 202 @ G^3v^4
- 307 @ G^4v^2
- 50 @ G^5

Jaranowski and Schafer 2012

PN vs NR

- All parameters are gauge dependent
- Parameters are defined differently

$$S^{\alpha\beta\mu}(\mathbf{x}) = M_0^{\alpha\beta\mu}(x) + x^\alpha T^{\beta\mu}(\mathbf{x}) - x^\beta T^{\alpha\mu}(\mathbf{x})$$

$$S^{\alpha\beta}(x) = \oint_{\partial\Omega} S^{\alpha\beta\mu} d\Sigma_\mu$$

$$\begin{cases} S_\mu^A = \epsilon_{\rho\sigma\nu\mu} S_A^{\rho\sigma} p_A^\nu / (2m_A c) \\ S_A^{\mu\nu} u_\nu^A = 0 \quad \text{(Spin supplementary condition)} \end{cases}$$

$$S_\phi \equiv \frac{1}{8\pi} \int_{\mathcal{H}} \phi^i s^j K_{ij}\, dA$$

$\mathcal{H}$: Apparent horizon

$$M = \oint_{\partial\Omega} T^{00} n^\mu d\Sigma_\mu$$

$$M^2 \equiv M_{\rm irr}^2 + \frac{S^2}{4M_{\rm irr}^2}, \quad M_{\rm irr}^2 \equiv \frac{A}{16\pi}$$

PN vs NR

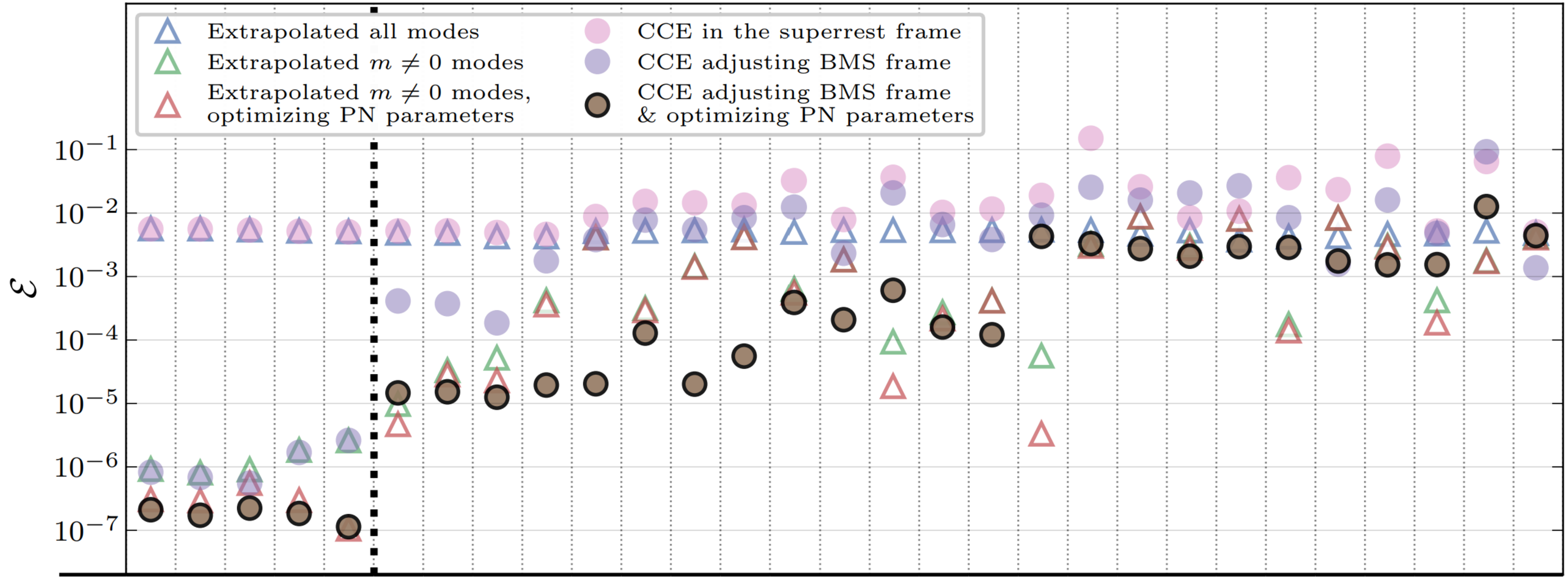
- Different asymptotic gauge (BMS frame)

PN	NR
Impose boundary at past null infinity	Determined by the Cauchy evolution Fix gauge using CCE/CCM

- Both have intrinsic error

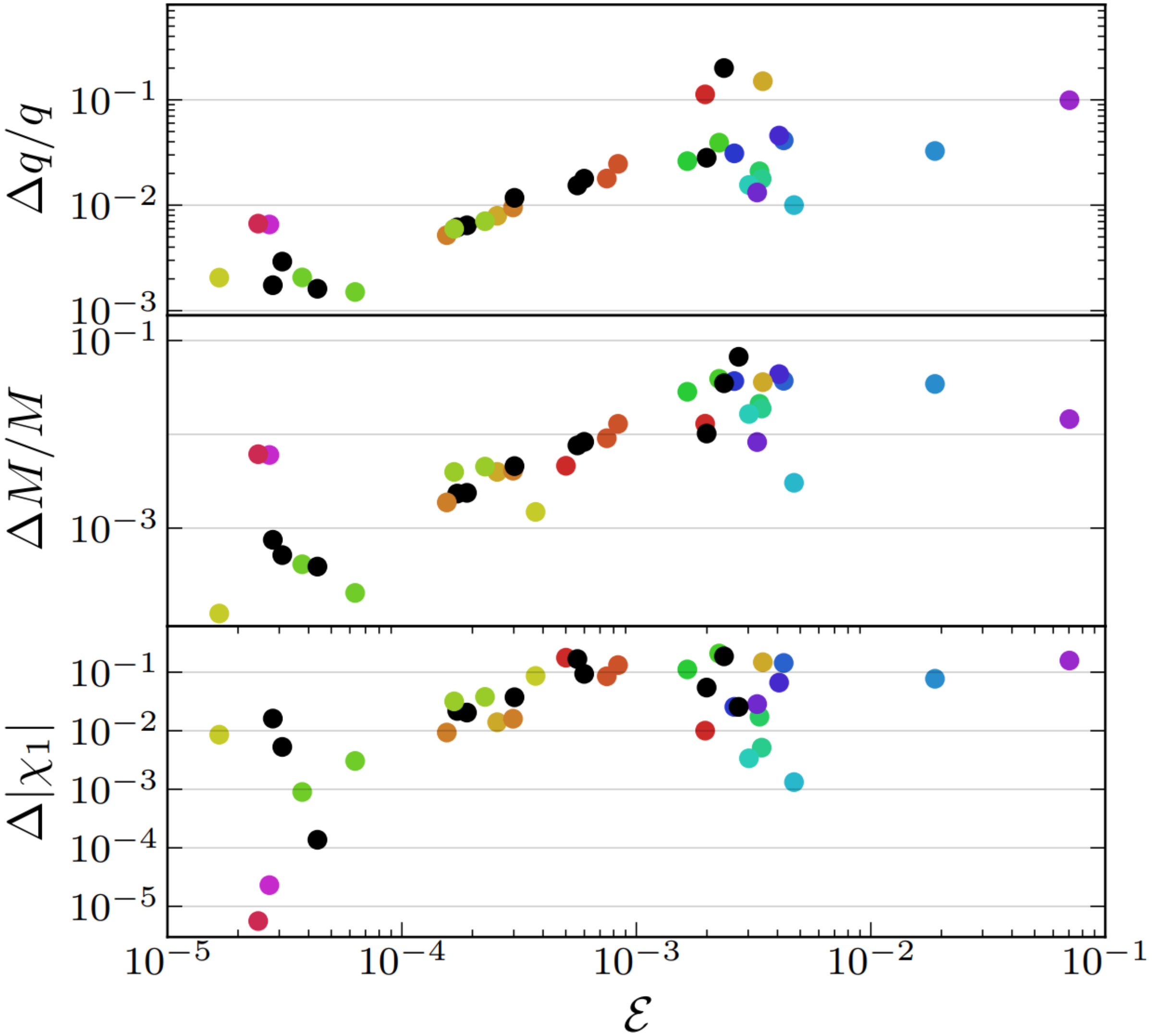
PN	NR
Expansion truncation error	Numerical truncation error

PN vs NR



SXS ID	1415	1413	1412	1414	1416	2637	2640	2639	2638	2622	2619	2617	2623	2625	2624	2626	2620	2618	2621	2629	2627	2628	2632	2633	2635	2634	2636	2631	2630
χ_p	0.0	0.0	0.0	0.0	0.0	0.05	0.05	0.05	0.08	0.32	0.33	0.36	0.4	0.4	0.43	0.45	0.46	0.46	0.49	0.5	0.5	0.5	0.56	0.56	0.57	0.57	0.57	0.8	0.8
q	1.5	1.4	1.6	1.8	1.8	8.0	8.0	8.0	8.0	1.5	1.8	1.6	1.9	4.0	1.2	4.0	1.5	1.4	1.8	10.0	6.0	8.0	4.0	8.0	8.0	4.0	8.0	8.0	4.0

PN vs NR



$\Delta q/q$
10^{-1}
10^{-2}
10^{-3}
$\Delta M/M$
10^{-1}
10^{-3}
$\Delta|\chi_1|$
10^{-1}
10^{-2}
10^{-3}
10^{-4}
10^{-5}
10^{-5}
10^{-4}
10^{-3}
10^{-2}
10^{-1}
$\mathcal{E}$


Careful about the parameters

Thank you!