Introduction to General-Relativistic Magnetohydrodynamics (GRMHD)

Jiaxi Wu

ARC seminar

Jan 31, 2025

What is and Why GRMHD?

GR M HD

Including GR effects, e.g. Strong gravity, compact objects, relativistic jets

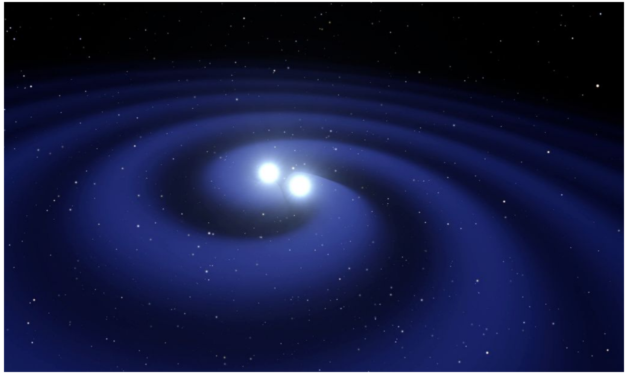
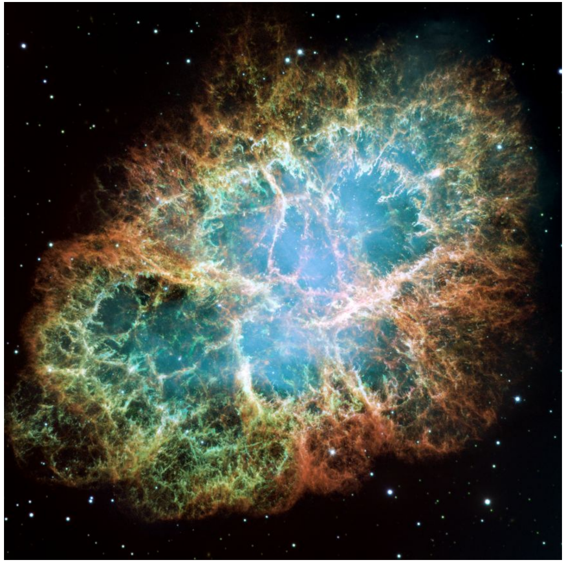
Including (often strong) magnetic fields across different scales

Including matter, i.e. astrophysical plasma (conducting fluid)


Electromagnetism
Fluid Dynamics
Magnetohydrodynamics

Where do we apply GRMHD?

- Accretion disks around black hole
- Relativistic jets and gamma-ray bursts
- Binary neutron star or neutron star-black hole mergers
- Supernovae or collapsar

Why we can model astrophysical plasma as fluid?

- The plasma is collisional, so that collective behavior of the plasma dominates
- The simulation scale is much larger than the microscopic scales of plasma
- Thermodynamic Equilibrium is established locally

Cases where MHD approximation fails

- Collisionless plasmas (e.g. solar wind, certain regions of ISM), we need to solve full Vlasov equations
- Small scales (e.g. magnetic reconnection), often studied with PIC simulations
- Ultra-strong magnetic fields, Turbulence, etc.

Let's start from MHD

Continuity equation (conservation of mass)

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0$$

Equation of motion (conservation of momentum)

$$\rho \left(\frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla \right) \mathbf{v} = \mathbf{J} \times \mathbf{B} - \nabla p$$

Induction equation

$$\frac{\partial \mathbf{B}}{\partial t} = \eta \nabla^2 \mathbf{B} + \nabla \times (\mathbf{v} \times \mathbf{B})$$

Equation of state (close the system)

Constraint of magnetic field

Now let's move to GR spacetime

Use covariant derivatives to rewrite MHD equations

Continuity equation (conservation of rest-mass)

Conservation of energy-momentum

$$\nabla_\mu(\rho u^\mu) = 0, \quad \nabla_\mu T^{\mu\nu} = 0.$$

Maxwell's equations

$$\nabla_\mu F^{\mu\nu} = \mathscr{I}^\mu, \quad \nabla_\mu {}^*F^{\mu\nu} = 0$$

To close the system, we need to specify the stress-energy tensor, and EoS.

$$T^{\mu\nu} := T_{\mathrm{m}}^{\mu\nu} + T_{\mathrm{f}}^{\mu\nu}$$

3+1 decomposition of spacetime

- The four-dimensional spacetime is splitted into 3D hypersurfaces + 1D timelike normal vector
- The metric is expressed in

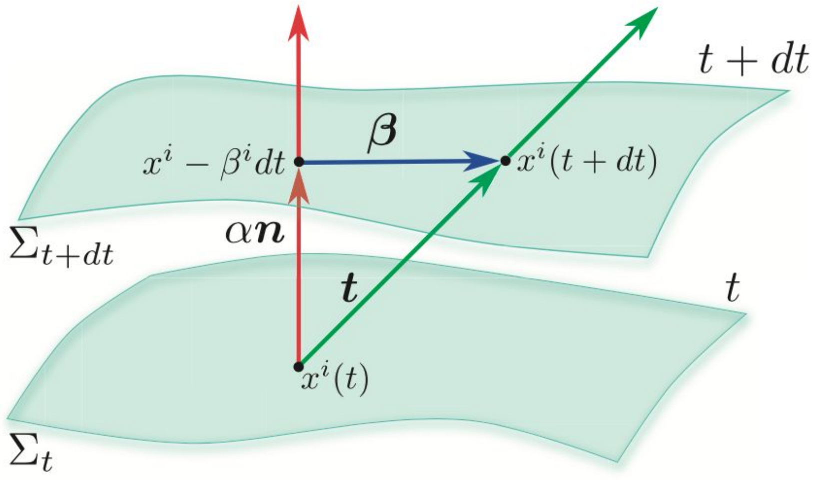
$$g_{\mu\nu} = \begin{pmatrix} -\alpha^2 + \beta_i\beta^i & \beta_i \\ \beta_i & \gamma_{ij} \end{pmatrix}$$

- The normal vector is

$$n_\mu = (-\alpha, 0, 0, 0)$$

$$n^\mu = \frac{1}{\alpha}(1, -\beta^i)$$

(we have gauge freedom)



Mizuno & Rezzolla

Eulerian frame and comoving frame

- In the 3+1 metric, tensor can be projected both parallel to the normal vector and on the hypersurface, e.g. 4-velocity

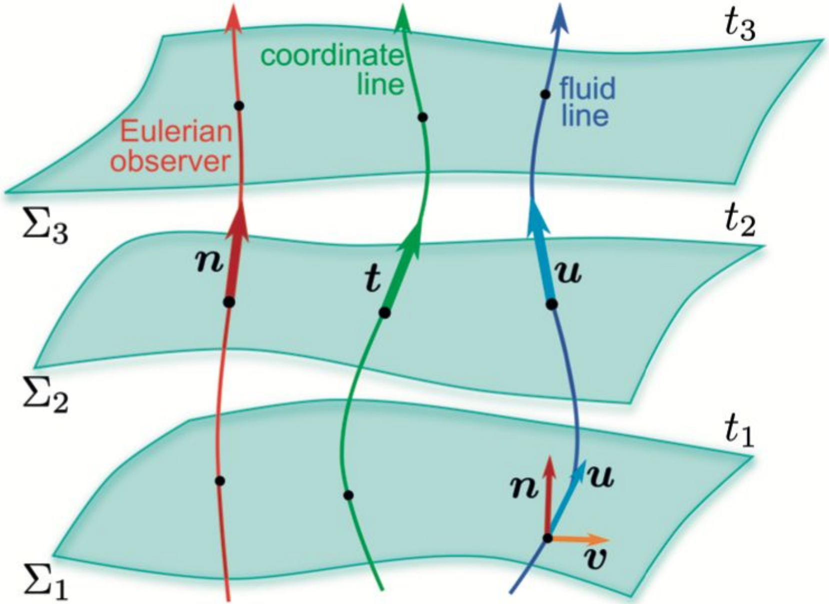
$$u^\mu = W(n^\mu + v^\mu)$$

Each physical quantity can be expressed

in Eulerian frame or in fluid comoving frame

$$F^{\mu\nu} = u^\mu e^\nu - u^\nu e^\mu - \varepsilon^{\mu\nu\lambda\delta} u_\lambda b_\delta$$

$$F^{\mu\nu} = n^\mu E^\nu - n^\nu E^\mu - \varepsilon^{\mu\nu\lambda\delta} n_\lambda B_\delta$$



Mizuno & Rezzolla

How to solve numerically?

- To solve the equations numerically, we have to make sure that they can be written into hyperbolic (i.e. well-posed) formalism.
- Generally, if the coefficient matrix of a set of conservative time-evolution equations is diagonalizable into a set of real eigenvalues, it is well-posed.

$$\partial_t \boldsymbol{U} + \boldsymbol{A} \cdot \boldsymbol{\nabla} \boldsymbol{U} = \boldsymbol{S}$$

- Furthermore, if the coefficient matrix is the Jacobian of some flux vector

$$\partial_t \boldsymbol{U} + \boldsymbol{\nabla} \boldsymbol{F}(\boldsymbol{U}) = \boldsymbol{S}$$

- The set of variables here are called conserved variables, which are nonlinear combination of primitive variables. We now try to rewrite the equations into such conserved formulation.

Conserved (Valencia) formulation

We use the continuity equations as an example

$$\nabla_\mu (\rho u^\mu) = \frac{1}{\sqrt{-g}} \partial_\mu \left(\sqrt{-g}\rho u^\mu\right)$$
$$= \frac{1}{\sqrt{-g}} \left[\partial_t \left(\sqrt{-g}\rho u^t\right) + \partial_i \left(\sqrt{-g}\rho u^i\right)\right] = 0$$

Use the 3+1 decomposition of 4-velocity $u^\mu = W(n^\mu + v^\mu)$

$$\partial_t(\sqrt{\gamma}\rho W) + \partial_i[\sqrt{\gamma}\rho W(\alpha v^i - \beta^i)] = 0$$

Define $D = \rho W$, we have

$$\partial_t \left(\sqrt{\gamma} D\right) + \partial_i \left[\sqrt{\gamma} D \left(\alpha v^i - \beta^i\right)\right] = 0$$

Conserved formulation (ideal MHD limit)

Similar to the continuity equation, we can write the conservation of energy-momentum, as well as induction equation into conserved formalism

$$\partial_t(\sqrt{\gamma}\boldsymbol{U}) + \partial_i(\sqrt{\gamma}\boldsymbol{F}^i) = \sqrt{\gamma}\boldsymbol{S}$$

$$\boldsymbol{U} = \begin{pmatrix} D = \rho W \\ S_j = DWhv_i + \epsilon_{ijk}E^j B^k \\ \tau = D(hW-1) - P + \frac{1}{2}(E^2 + B^2) \\ B^k \end{pmatrix} \qquad \boldsymbol{F}^i = \begin{pmatrix} D\tilde{v}^i \\ S_j\tilde{v}^i + p^*\delta^i_j - b_j B^i/W \\ \tau\tilde{v}^i + p^* v^i - \alpha b^0 B^i/W \\ \tilde{v}^i B^k - \tilde{v}^k B^i \end{pmatrix}$$

$$\boldsymbol{S} = \begin{pmatrix} 0 \\ \frac{1}{2}\alpha W^{ik}\partial_j\gamma_{ik} + S_i\partial_j\beta^i - (D+\tau)\partial_j\alpha \\ \frac{1}{2}W^{ik}\beta^j\partial_j\gamma_{ik} + W^j_i\partial_j\beta^i - S^j\partial_j\alpha \\ 0 \end{pmatrix}$$

Don’t forget the convergence-free constraints on magnetic fields.

Couple to spacetime evolution

Z4c (3+1 decomposition of Einstein's equation + conformal transformation)

$$\partial_t[\chi] = \frac{2}{3}\chi(\alpha(\hat{K} + 2\Theta) - \partial_i[\beta^i]) + \beta^i\partial_i[\chi],$$

$$\partial_t[\tilde{\gamma}_{ij}] = -2\alpha\tilde{A}_{ij} + \beta^k\partial_k[\tilde{\gamma}_{ij}] - \frac{2}{3}\tilde{\gamma}_{ij}\partial_k[\beta^k] + 2\tilde{\gamma}_{k(i}\partial_{j)}[\beta^k].$$

$$\partial_t[\hat{K}] = -D^i[D_i[\alpha]] + \alpha\left[\tilde{A}_{ij}\tilde{A}^{ij} + \frac{1}{3}(\hat{K} + 2\Theta)^2\right] + \beta^i\partial_i[\hat{K}] + \alpha\kappa_1(1 - \kappa_2)\Theta + 4\pi\alpha[S + \rho],$$

$$\partial_t[\tilde{A}_{ij}] = \chi[-D_i[D_j[\alpha]] + \alpha(R_{ij} - 8\pi S_{ij})]^{\mathrm{tf}} + \alpha[(\hat{K} + 2\Theta)\tilde{A}_{ij} - 2\tilde{A}^k{}_i\tilde{A}_{kj}] + \beta^k\partial_k[\tilde{A}_{ij}] + 2\tilde{A}_{k(i}\partial_{j)}[\beta^k] - \frac{2}{3}\tilde{A}_{ij}\partial_k[\beta^k],$$

$$\partial_t[\Theta] = \frac{\alpha}{2}[\tilde{\mathcal{H}} - 2\kappa_1(2 + \kappa_2)\Theta] + \beta^i\partial_i[\Theta],$$

$$\partial_t[\tilde{\Gamma}^i] = -2\tilde{A}^{ij}\partial_j[\alpha] + 2\alpha\left[\tilde{\Gamma}^i{}_{jk}\tilde{A}^{jk} - \frac{3}{2}\tilde{A}^{ij}\partial_j[\ln(\chi)] - \kappa_1(\tilde{\Gamma}^i - \hat{\Gamma}^i) - \frac{1}{3}\tilde{\gamma}^{ij}\partial_j[2\hat{K} + \Theta] - 8\pi\tilde{\gamma}^{ij}S_j\right] + \tilde{\gamma}^{jk}\partial_k[\partial_j[\beta^i]] + \frac{1}{3}\tilde{\gamma}^{ij}\partial_j[\partial_k[\beta^k]] + \beta^j\partial_j[\tilde{\Gamma}^i] - \hat{\Gamma}^j\partial_j[\beta^i] + \frac{2}{3}\hat{\Gamma}^i\partial_j[\beta^j];$$

$$\partial_t[\alpha] = -\mu_L\alpha^2\hat{K} + \beta^i\partial_i[\alpha],$$

$$\partial_t[\beta^i] = \mu_S\alpha^2\tilde{\Gamma}^i - \eta\beta^i + \beta^j\partial_j[\beta^i]$$

Numerical implementation

- Several codes have been implemented to solve GRMHD equations numerically, e.g. Athena++/AthenaK, IllinoisGRMHD, WhiskyMHD
- During the initialization stage, the code typically read data from external initial data library, e.g. Lorene for BNS merger. The initial data should satisfy the constraints both for spacetime evolution and MHD evolution.
- General steps during one time-evolution step
  - Update metric variables and enforce constraints
  - Reconstruct variables and solve the Riemann problem at cell interfaces
  - Add source terms and update conservative variables
  - Enforce divergence-free condition
  - Apply primitive recovery and flooring schemes
  - Compute the stress-energy tensor and feedback into Einstein’s equations.

Numerical scheme—Reconstruction

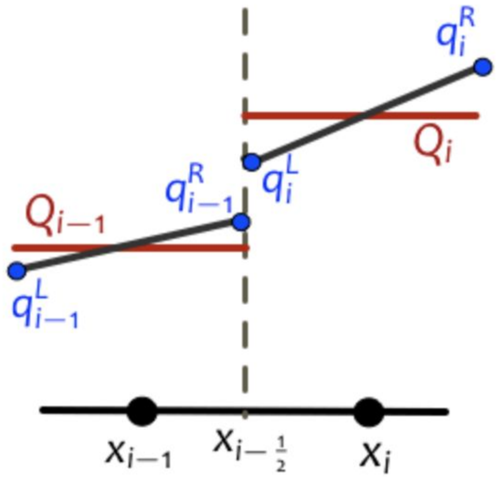
Goal: Given cell-centered value, construct high-order accurate "left" and "right" states at each cell interface.

- Widely used methods: PLM (piecewise linear) and PPM (piecewise parabolic)
- Key steps:
  - Compute (limited) slopes within each cell
  - Construct a polynomial (linear, parabolic, or higher) approximation inside the cell
  - Evaluate the polynomial at the left and right faces of the cell
- Difficulties:
  - Near shocks/discontinuities, reconstruction might degrade to first-order accuracy, and limiter must be designed so that no spurious oscillations would happen
  - Near vacuum floors might render large errors
  - Coordinate singularities and mesh refinement boundaries need special treatment

Numerical Scheme—Riemann solver

Goal: At each cell interface, given the reconstructed left and right states, solve the local Riemann problem to compute fluxes across the interface.

- Algorithms: HLLE, LLF, WENO-Z, etc. Often approximate solvers
- Key steps:
  - Calculate/approximate the wave speed of different modes
  - Calculate the numerical fluxes from left and right states, as well as the wave speeds
- Difficulties:
  - Complex wave structure in GRMHD (fast, slow, Alfven, etc.) which is hard to distinguish
  - Strong gravitational fields might cause extreme wave speed
  - Exact GRMHD Riemann solver is prohibitively complicated, approximate solvers do not track each wave exactly



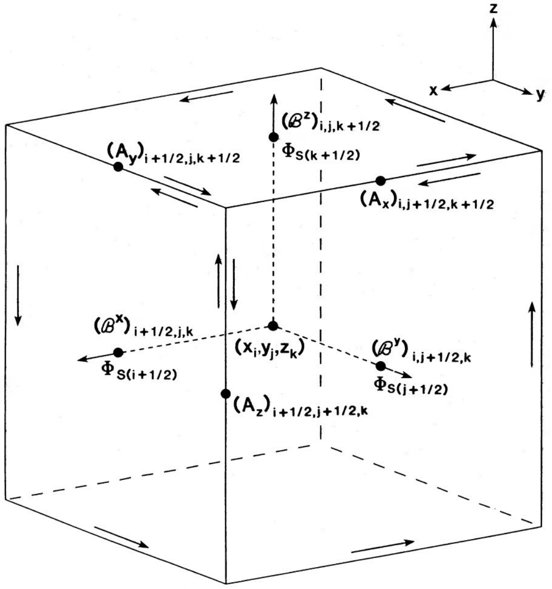
Numerical scheme—Constrained transport

Goal: Numerically evolve the magnetic field such that the divergence-free constraint is guaranteed on machine-precision level.

- From Faraday's law $\frac{d}{dt}\int_{\text{face}} \mathbf{B}\cdot d\mathbf{A} = -\oint_{\partial(\text{face})} \mathbf{E}\cdot d\mathbf{l}$
- This automatically guarantees the divergence-free constraint of B field

$$\oint_{\partial V} \mathbf{B}\cdot d\mathbf{A} = 0$$

- In this case, we stagger the value of magnetic field to face center, and reconstruct and calculate the numerical flux on face interface, i.e. edges



Evans & Hawley,1988

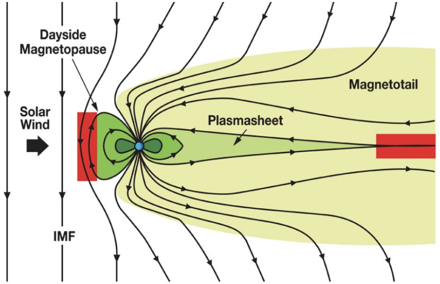
Numerical scheme—Primitive recovery

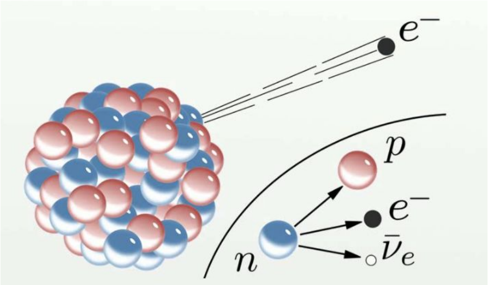
Goal: Compute primitive variables (which are then used to calculate fluxes using a Riemann solver) from conservative variables

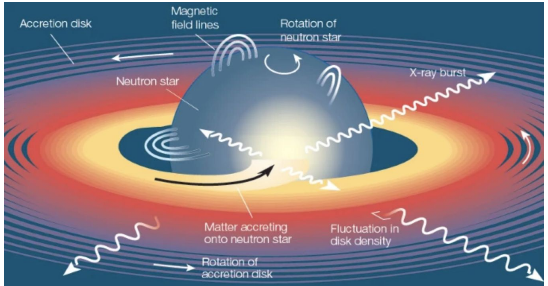
- Cons. are nonlinear functions of prim., so iterative root-finding methods are often used
- Specialized methods can be used to reduce number of unknowns, or effectively bracket the root
- Tabulated nuclear physics equation of state adds more complexity to the root-finding process
- Effective flooring schemes need to be implemented to avoid unphysical results

Add more physics?

- Beyond the ideal MHD limit, we want to incorporate accurate microphysics
  - Resistivity (magnetic reconnection, laboratory plasmas)
    - Add diffusion terms to the induction equation
  - Viscosity (accretion disks, differential rotation, turbulence)
    - Add viscous stress tensor to momentum equation
  - Neutrino physics (core-collapse supernovae, neutron star mergers)
    - Add stress-energy tensor of neutrinos, often captured by leakage scheme
  - Finite-temperature EoS (shock heating, SNe cores, BNS merger)
  - Radiation transport (gamma-ray bursts, super-eddington flows, AGN)
    - Add stress-energy tensor of radiation field







Current limitations

- Extreme spatial scales and AMR complexity
- Microphysics and complex equations of state
- CPU/GPU parallel scalability
- Uncertain and limited initial conditions
- Boundary conditions and outflows
- Turbulence at small scale and reconnection

Subgrid dynamo

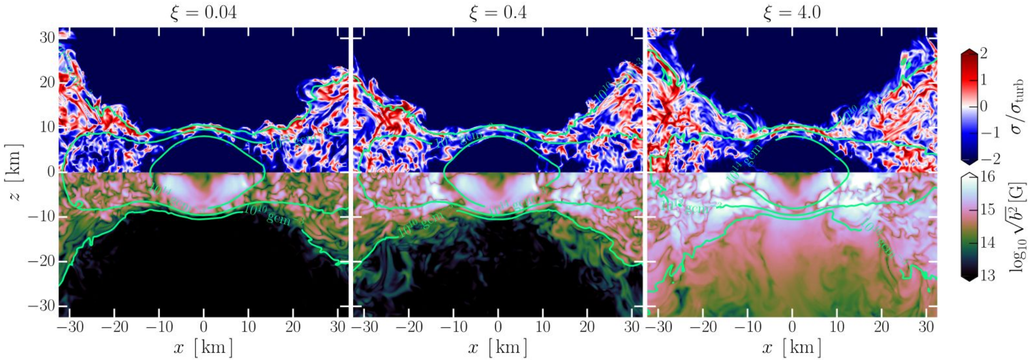
Goal: effectively simulate the evolution and amplification of magnetic field during astrophysical processes without going to ultra-high resolution

- Mean field dynamo

$$\bar{E}^i = -\varepsilon^{ijk}\bar{v}_j\bar{B}_k - \langle \delta\mathbf{v} \times \delta\mathbf{B} \rangle^i$$

$$\langle \delta\mathbf{v} \times \delta\mathbf{B} \rangle^i \approx \alpha^i_j \bar{B}^j - \beta^i_l \varepsilon^{ljk}\partial_j \bar{B}_k$$

$$\approx \alpha^i_j \bar{B}^j - \beta^i_k J^k$$



Most, 2023

Summary and Takeaways

- GRMHD combines the principles of general relativity, magnetohydrodynamics, and astrophysical plasma physics
- GRMHD simulations are widely used in systems such as black hole accretion disks, relativistic jets and gamma-ray bursts, binary neutron star mergers, and core-collapse supernovae
- The numerical stability requires well-posed hyperbolic systems
- Several advanced numerical schemes are implemented, including reconstruction, Riemann solver, constrained transport and primitive recovery
- Extreme spatial and temporal scales demand further computational techniques as well as hardware development
- More accurate model of microphysics is needed